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Viable Techniques, Leontief's Closed Model, and Sraffa's Subsistence Economies

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ABSTRACT

This paper studies the production techniques employed in economies that reproduce themselves. Special attention is paid to the distinction usually made between those that do not produce a surplus and those that do, which are referred to as first and second class economies, respectively. Based on this, we present a new definition of viable economies and show that every viable economy of the second class can be represented as a viable economy of the first class under two different forms, Leontief's closed model and Sraffa's subsistence economies. This allows us to present some remarks concerning the economic interpretation of the two models. On the one hand, we argue that the participation of each good in the production of every good can be considered as a normal characteristic of the first model and, on the other hand, we provide a justification for the same condition to be considered a characteristic of the second model. Furthermore, we discuss three definitions of viable techniques advanced by other authors and show that they differ from ours because they admit economies that do not reproduce them selves completely.

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1.0 INTRODUCTION

In single production models, in which each industrial branch produces exclusively one type of good, one of the basic research topics is the definition of the conditions that a given technique has to satisfy in order to support a production program that reproduces itself.[3] That is to say, one which produces at

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[3] A production program is the ordered set of data, usually presented in matrix notation, indicating the amount of each good produced and also the amounts of each good and of labor consumed directly in each industrial branch during the time considered. For the purposes of this article, we will call technique a production program that produces a unit of each good and therefore it also indicates the amounts of each good and of labor consumed per unit of good produced. In reference to

least the amount of each good consumed and where, in addition, a price system exists that allows each industry to recover at least the amount invested. For such a study, it is useful to consider separately, among programs reproducing themselves, those that do not produce a surplus from those that do, which will be referred to as first and second class programs, respectively.

The closed model introduced by (Leontief, 1960) and the subsistence economies defined by (Sraffa, 1960) have attracted most interest among first class programs. However, we identify in the specialized literature two issues that deserve further consideration. Both are related to the condition that the matrix of technical coefficients is indecomposable, which is equivalent to the condition that each good produces every good or, to be brief, that every good is basic.[4] On the one hand, this condition is adopted as a simplifying assumption in some studies on Leontief's closed model (e.g., Dorfman, Samuelson & Solow, 1986, 254-264; Wurtele, 1960, 25) or, as we argue, it can also be considered as a general characteristic of the model. On the other hand, the same condition is adopted by (Sraffa, 1960, 7) in his definition of subsistence economies but he provided no justification. We discuss these two topics and show that each one of these models may be understood as a special form to represent second class economies. From this perspective, if the second class economy satisfies the condition established by (Hawkins & Simon, 1949), usually referred to as (H-S), each good is basic in the first case as a consequence of the link between production and consumption provided by labor and, in the second case, as a consequence of the link between prices and national income provided by the value added in each one of the industrial branches.

These, and other results presented ahead, suggest defining viable techniques as those where the corresponding coefficient matrix satisfies either condition (H-S) or the two conditions of being indecomposable and having its Frobenius root equal to one. The first condition and the last two characterize the viable techniques employed respectively in second and first class economies. On this basis, we say that an economy is viable if it uses a viable technique. Although the mathematics involved in this definition are well known, several other definitions have been proposed in the specialized literature and, for this reason, we make a few comparative studies.

Including this introduction, the article is divided in seven sections. Section 2.0, presents a brief survey of the relevant literature. Section 3.0, presents the reference model and a mathematical characterization of viable techniques. On Section 4.0, we show that every viable economy of the second class may be represented as a Leontief's closed model where every good is basic. In addition to this result, we present some other arguments for the thesis that Leontief's closed model may be considered as one that normally satisfies this condition. On Section 5.0, we show that any viable economy of the second class may be represented as a Sraffa's subsistence economy, which provides a justification for the assumption that every good is basic in that model. Section 6.0, considers three alternative definitions of viable economies used on the literature on linear production systems with the purpose of throwing light on some features that set them apart from the one adopted in this article. On the last section, we present some comments of a general character.

2.0 A BRIEF SURVEY OF THE LITERATURE

With respect to technique in general, the following two problems are proposed (Takayama, 1985, 360), the first one has to do with the physical aspect and the second one is related to financing of the reproduction:

> *What conditions must be satisfied in order to sustain an economy producing the means of production consumed plus a given physical surplus?* (1)

the last mentioned programs, ten Raa (2005, 55) employs the expression *pure* technique. Following common usage, we will also use the term economy to refer to a production program.

[4] We say that a good *j* produces a good *i* if *j* contributes to the production of *i* either directly or indirectly, as explained in Section 3.0 ahead.

What conditions must be satisfied in order to sustain an economy for which a unique price system exists allowing each branch to recover the investment made plus a given share of the national income? (2)

According to (Kurz & Salvadori, 1995, 381), these questions are already clearly visible in the works of early economists such as (Ricardo, 2004) and (Torrens, 1972). (Dmitriev,1974) and (Cassel, 1967) offer some of the first representations of single production economies by means of systems of linear equations, which constitute their usual representation in contemporary works starting with (Leontief, 1960). Assuming constant returns to scale, the viability conditions mentioned in the previous section answer these questions as they relate to the matrix of technical coefficients of means of production - those corresponding to labor are not required.[5] The couple of conditions for the first class and the condition for the second class are necessary and sufficient to satisfy questions (1) and (2) on each. Moreover, on economies of the first class, the price system is entirely positive and it is determined up to a scalar multiple while on economies of the second class, given any distribution of national income between the different industrial branches, the price system is entirely positive and univocally determined.

There is general agreement among researchers regarding condition (H-S) as characteristic of second class viable techniques. However, the couple of conditions mentioned above is not similarly accepted for first class viable techniques. Indeed, it is generally assumed that one of the eingenvalues of the coefficients matrix equals one, which corresponds to the fact that, in economies of the first class, the quantity consumed of each good is equal to the quantity produced of that good. Thus, at least one good is employed as a means of production. Furthermore, the coefficients matrix is indecomposable if and only if:

Every good is basic. (3)

Regarding this conclusion, as already indicated, some authors adopt it as a simplifying assumption while others consider it to be necessary without providing a justification.[6] There is a very extensive literature on single production models and a detailed exposition can be found in the work by Kurz & Salvadori already cited. This last work, and also (Bidard, 2004), and (Pasinetti, 1977), offers the definitions of viable economies discussed further in this paper. Regarding the mathematical conditions that viable economies satisfy, the interested reader may find ample information in (Takayama, 1985), as well as in (Gantmatcher, 1960) and (Nikaido, 1970).

3.0 VIABLE TECHNIQUES

In this Section, we present the model of single production economy under study. We also indicate the viability conditions for techniques employed in economies of the first and the second class and, finally, we formulate and discuss our definition of viable techniques.

3.01 THE MODEL

The reference economy is integrated by n ($n \geq 1$) industries, each one producing a particular type of good labeled i or j so that $i, j = 1, 2, \ldots, n$. We will refer to a set $\{j1, j2, \ldots, jD\}$ as a D-set if it contains D different goods and, for any particular D-set, we have $d = 1, 2, \ldots, D$. For each pair (i, j) and for each i, a_{ij}

[5] Under this assumption, given a bundle of goods, the program producing the bundle with a particular technique results from multiplying the data corresponding to each industry times the amount of goods to be produced. Conversely, given a production program, the technique used results from dividing the data of each industry by the amount of goods that it produces.

[6] As shown in Section 4.02 ahead, the fact that one of the eingenvalues of the coefficient matrix is equal to one, together with condition (3), imply that the Frobenius root of the matrix is equal to one.

and l_i are respectively the quantity of *j* and of labor consumed directly in the production of one unit of *i*. Regarding these technical coefficients we adopt the following assumptions.

$$\text{a)}\quad a_{ij} \geq 0 \text{ for every } (i, j) \qquad\qquad \text{b) } l_i > 0 \text{ for every } i \tag{4}$$

A good *j* produces a good *i* (not necessarily different) either directly if $a_{ij} > 0$ or indirectly if there is a D-set containing neither *i* nor *j* and verifying $a_{i,j1}a_{j1,j2}a_{j2,j3}\ldots a_{jDj} > 0$. Furthermore, for each *i*, we denote p_i, z_i, x_i and c_i respectively the price of good *i*, the sum of wages and profits corresponding to branch *i* per unit of good,[7] the quantity of *i* produced in the corresponding industry and the difference between this quantity and the amount of the same good that is consumed in the industrial system during the period.

The first of the following equations systems represents the relations between the quantities consumed and produced of the different goods. The second one represents the relation between each price and the corresponding production cost.

$$\text{a)}\sum_i a_{ij}x_i + c_j = x_j \quad j = 1, 2, \ldots, n \qquad \text{b)}\sum_j a_{ij}p_j + z_i = p_i \quad i = 1, 2, \ldots, n \tag{5}$$

It is useful to write these systems in matrix notation. To this end, we define the column vectors $x = (x_1, x_2, \ldots, x_n)^T$, $c = (c_1, c_2, \ldots, c_n)^T$, $p = (p_1, p_2, \ldots, p_n)^T$ and $z = (z_1, z_2, \ldots, z_n)^T$, together with the input matrix $A = [a_{ij}]$. Then, we can write the preceding systems as follows:

$$\text{a) } A^T x + c = x \qquad\qquad \text{b) } Ap + z = p \tag{6}$$

The Frobenius roots of matrices *A* and A^T, which are equal, are represented with λ_A. A square matrix $A \geq 0$ may be interpreted as an input matrix corresponding to an economy that produces one unit of each good.[8] Assuming this interpretation, and in order to simplify, we will refer to any such matrix as a technique even if the labor amounts are not indicated.

3.02 VIABILITY CONDITIONS FOR ECONOMIES OF THE FIRST CLASS

In economies of the first class, $c = z = 0$. Due to these facts, the following results are important with regard to questions (1) and (2) in this case.

Theorem 1. *Let A be a square matrix such that $A \geq 0$. Consider the propositions:*
- (i) *There is a solution $x > 0$ and $p > 0$ respectively to equations (6.a) and (6.b).*
- (ii) *If there is a solution $x' \geq 0$, $x' \neq 0$ or $p' \geq 0$, $p' \neq 0$ respectively to equation (6.a) or (6.b), then $x = \theta x'$ or $p = \theta p'$ for some positive scalar θ.*

When $c = z = 0$, both propositions are true if and only if A satisfices the following conditions:

$$\text{a) } A \text{ is indecomposable} \qquad\qquad \text{b) } \lambda_A = 1 \tag{7}$$

Proof. See the Appendix.

Therefore, (7) indicates a pair of necessary and sufficient conditions answering (1) and (2) in the case of techniques used in economies of the first class. It is worth remarking that (ii) of Theorem 1 means that,

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[7] Usually, a value added coefficient (z_i) is the difference between the revenues per unit of output (the price of the commodity) and the material cost per unit of output (ten Raa, 2005, 19).

[8] Given two matrices (*A*, *B*) or two vectors (*x*, *y*), the relations *A* = *B* and *x* = *y* means respectively that $a_{ij} = b_{ij}$ for every couple (*i*,*j*) and $x_j = y_j$ for every *j*. We define each one of the relations ">", "<", "≥" and "≤" in a similar manner while the relation "≠" means that "=" is not true. If all the entries of a matrix or a vector are equal to zero we may represent it with 0.

if the conditions are satisfied, relative prices are determined although the unit in which prices are measured has yet to be defined. Similarly, the proportions between the quantities of goods to be produced are fixed but not the quantities.

3.03 VIABILITY CONDITIONS FOR ECONOMIES OF THE SECOND CLASS

In economies of the second class, $c \neq 0$ and $z \neq 0$. For this reason, the following proposition is important with regard to questions (1) and (2) in this case.

Proposition 1. *For any given vector $c \geq 0$, $c \neq 0$ or $z \geq 0$, $z \neq 0$ there is a unique solution $x \geq 0$, $x \neq 0$ or $p \geq 0$, $p \neq 0$ respectively to (6.a) and (6.b).*

Let $B = [I - A]$, where I is the identity matrix. According to Theorem 4.D.2 by (Takayama, 1985, 392), Proposition 1 is valid if and only if any of the following conditions is satisfied.

a) (H-S): *all the successive principal minors of B are positive*. b) $0 \leq \lambda_A < 1$. (8)

The distribution of coordinates greater than zero and zero in the solution to system (6.a) depends on the participation of the different goods in the production process. In fact, coordinates greater than zero correspond exclusively to the goods that are part of the surplus and to those that produce such goods (Benítez, 2009, 94). It is worth noting that, following a procedure similar to the one employed in the proof of this result, it is also possible to prove a similar proposition for equation (6.b). In other words, if A satisfies (8), given a vector $z \geq 0$, $z \neq 0$, prices greater than zero correspond exclusively to goods produced in industries were the sum of wages and profits is greater than zero and to goods used directly or indirectly in those industries.

Therefore, in what follows we will assume that, in the production programs considered, an amount greater than zero of each good is produced without loss of generality in the analysis. Indeed, if a production program that reproduces itself produces certain goods and not others, it is possible to substitute it for the program consisting exclusively of those branches of production whose production is not zero.

According to the preceding analyses, the following conclusion is true in economies of the second class satisfying (8).

Every good either is produced in surplus or produces at least one good produced in surplus, or both. (9)

3.04 A DEFINITION OF VIABLE TECHNIQUES

The preceding results suggest the following definition:

Definition 1. *Let A be a square matrix such that $A \geq 0$. A is a viable technique if it satisfies either (7) or (8).*

As a consequence, viable techniques are those that have the necessary and sufficient conditions responding to questions (1) and (2). It is worth noting that this definition includes the limit case $A = 0$, which represents an economy that produces one unit of each good without consuming any production means.

To illustrate certain aspects of Definition 1, let us consider the following examples.

$$\text{a) } \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \qquad \text{b) } \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{c) } \begin{bmatrix} 1/7 & 0 \\ 0 & 1/6 \end{bmatrix} \tag{10}$$

Matrix (10.a) and (10.b) are not viable because, although both satisfy condition (7.b), they fail to satisfy (7.a). In the first case, the second industry is self-sufficient in good 2, but it produces no surplus to give in exchange for its requirement of good 1. In the second case, A does not correspond to a viable technique but to a technique composed of two independent viable techniques. In turn, matrix (10.c) can be said alternatively to correspond either to two independent viable techniques or to a single viable technique. The first interpretation can be adopted more likely if wages are paid entirely in kind because there is no apparent relation between the two industrial branches, while the second interpretation is appropriate when the wage is paid at least partially in value. Indeed, in this case labor and labor's share in the net product establish a relation between all branches in the industrial system, a matter studied in more detail in (Benítez & Benítez, 2014). Moreover, as we show in the next section, a similar relation may be provided by the net product and labor and, as we show in Section 5.0, also by the net product and the distribution of value added between the industrial branches.

4.0 LEONTIEF'S CLOSED MODEL

In this section we compare, on the one hand, Leontief's closed model and, on the other hand, the economies of the first and second classes. Our conclusions support the thesis that Leontief's closed model can normally be considered as a viable technique of the first class.

4.01 SECOND CLASS ECONOMIES AND LEONTIEF'S CLOSED MODEL

Given a viable production program of the second class, for each $i, j \le n$, let:

$$\text{a) } a_{i,n+1} = l_i \qquad \text{b) } a_{n+1,j} = c_j / \sum_j x_j l_j \tag{11}$$

Assumption (4.b) must be remembered regarding these coefficients and also the fact that $c_j > 0$ for at least one j. Hence, the following conditions are satisfied:

$$\text{a) } a_{i,n+1} > 0 \text{ for each } i \le n \qquad \text{b) } a_{n+1,,j} > 0 \text{ at least for one } j \le n \tag{12}$$

We can use the information from the program to form the following matrix:

$$Æ = \begin{bmatrix} A & L \\ C & 0 \end{bmatrix}$$

In which A is the matrix of means of production coefficients, $C = [a_{n+1,1} \quad a_{n+1,2} \ldots a_{n+1,n}]$ and L is the matrix defined by $L^T = [a_{1,n+1} \quad a_{2,n+1} \ldots a_{n,n+1}]$. Each row i of matrix Æ indicates inputs consumed by industry i, and each column j indicates quantities of good j consumed by the different industries. Regarding this, special attention is due to column $n + 1$ and row $n + 1$ of Æ because they correspond respectively to the set of households considered by Leontief as a particular industry whose product is work and whose inputs are the goods produced in surplus. He assumes that households do not use labor, which implies the following condition:

$$a_{n+1,n+1} = 0 \tag{13}$$

This representation of the economy is known as Leontief's closed model.[9] Let I be the identity matrix of order $n + 1$. By construction, the equations:

$$\text{a) } Æ^T x = x \qquad \text{b) } [I - Æ^T]x = 0 \tag{14}$$

[9] We must point out that condition (13) is not always stated explicitly.

have a solution, the vector x of order $(n + 1) \times 1$ in which each one of the first n coordinates equals the quantity of the corresponding good produced in the economy of reference while the $n + 1$ coordinate is the amount of work used by it. Because any multiple of x also satisfies (14), given the matrix Æ we can calculate the quantity of each good that must be produced as a function of the volume of any of the goods. For instance, if the amount of work used is multiplied times a positive constant ϑ, vector ϑx indicates the quantity of each good that must be produced.

It is important to remark that Æ is a viable technique of the first class. On the one hand, given that in the viable economies of the second class (9) is true, within the first n goods, if j is produced in surplus $a_{n+1,\ j} > 0$ and, if it is used as a means of production, j participates either directly or indirectly in the production of at least one good produced in surplus. In consequence, each of the first n goods participates in the production of good $n + 1$. Furthermore, according to (12.a), this last good participates directly in the production of each and every one of the first n goods. Therefore, all goods are basic, which implies that Æ is indecomposable. On the other hand, making $\lambda_{Æ} = 1$, we can also write equation (14.a) in the form $Æ^T x = \lambda_{Æ} x$. Due to the fact that there is a solution $x > 0$ for this equation, and also because Æ is indecomposable, it follows from (iv) of Theorem 4.B.1 by (Takayama, 1985, 372) that $\lambda_{Æ}$ is the Frobenius root of Æ. These results enable us to establish the following proposition.

Proposition 2. *Every viable economy of the second class allows the construction of a Leontief's closed model representing that economy with the resulting closed model being a viable technique of the first class.*

4.02 LEONTIEF'S CLOSED MODEL AS A FIRST CLASS ECONOMY

When studying Leontief's closed model, a square matrix $Æ \geq 0$ of order greater than one is always considered and, in addition to (4.a) and (13), the following assumption is usually adopted (e.g., Allen, 1966, 360; Leontief, 1960, 46-47; Pasinetti, 1977, 56):

$$\textit{The determinant of matrix } [I - Æ] \textit{ is equal to zero.} \tag{15}$$

This condition implies that one of the eigenvalues of Æ equals one. Because of this, $Æ \neq 0$ and, also, a vector x exists such that $x \geq 0$ and $x \neq 0$ which satisfies equations (14) (Takayama, 1985, 367). Nevertheless, (4.a), (13) and (15) are not sufficient for all semipositive solutions x to make sense economically. Let us consider, for example, the following techniques $Æ^T$:

$$\text{a)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 1/4 \\ 0 & 1/2 & 0 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Vector $(1, 0, 0)$ is the only semipositive solution, determined up to a scalar multiple, for equations (14) corresponding to matrix a). Quite differently, equations (14) corresponding to matrix b) are satisfied by every semipositive vector in which $x_3 = 0$. In each case, at least one good is produced destined entirely to its own reproduction without any labor participation. Moreover, in the second case the relative price between goods 1 and 2 is not determined by the corresponding system (6.b). In fact, any exchange relation between the two goods satisfies (6.b).

It is possible to avoid these difficulties if, in addition to (4.a), (13) and (15), we also assume conditions (9) and (12). Indeed, as already shown, (9) and (12.a) imply that Æ is indecomposable while (15) implies that one of the eigenvalues of Æ equals one. Given these results, it follows from (iv) of Theorem 4.B.1 by (Takayama, 1985, 372) that its Frobenius root equals one. This allows the following conclusion.

Proposition 3. *If in addition to the usual assumptions (4.a), (13) and (15), conditions (9), and (12) are also assumed, Leontief's closed model is a viable technique of the first class.*

Hence, the supplementary hypotheses proposed simplify the formal properties of the model. Furthermore, their adoption does not exclude from it any significant cases. It is worth adding that the simplification alluded does not depend on condition (13) and also, in this regard, that domestic labor may be represented by a coefficient $a_{n+1,n+1} > 0$.

From a different angle, if the conditions of Proposition 3 are satisfied, the inequality $\lambda_A < 1$ is true. Indeed, λ_A equals the Frobenius root of the matrix resulting from substituting in Æ each of the coefficients of row $n + 1$ and column $n + 1$ by zeros. It follows from (vi) of Theorem 4.B.1 by (Takayama, 1985, 372), (v) of Theorem 4.B.2 by (Takayama, 1985, 375), and from the equation $\lambda_{Æ} = 1$ that the root mentioned is smaller than one. And so, the following conclusion can be stated.

Proposition 4. *Every production program determined by a Leontief's closed model satisfying (4.a), (9), (12), (13) and (15) can also be represented as a viable economy of the second class.*

5.0 SRAFFA'S SUBSISTENCE ECONOMIES

By definition, Sraffa's subsistence economy is a viable economy of the first class. On this section, we will show that such model constitutes a particular form of representing a viable economy of the second class. Also, we will show that every viable economy of the first class represents at least one viable economy of the second class.

5.01 THE NATIONAL INCOME

For certain purposes, it is useful to define the technique employed in a production program by using the quantity produced of each good as a measurement unit of the quantities of that good within the program, as indicated next.

Definition 2. *The technical form of a production program is the program that results from dividing, on the one hand, the quantities of each good by the quantity produced of the good, and, on the other hand, the quantities of labor by the sum of the labor used in the program.*

According to this, if a production program reproducing itself is represented in its technical form, the following is true.

$$\text{a)} \sum_i a_{ij} \leq 1 \ \forall j \qquad \text{b)} \sum_i l_i = 1 \tag{16}$$

Furthermore, each coefficient a_{ij} indicates at the same time the quantity of j consumed per unit of i produced, and the proportion of the quantity produced of j that is consumed on industry i. In what fallows, we will assume that the production programs being considered are in their technical form. In this manner, we may speak of a square matrix A such that $A \geq 0$, as a technique or an economy without needing to make a distinction.

As already indicated, in a viable economy of the second class, for any given $z > 0$ there is a unique $p > 0$ satisfying (6.b). Summing up the n equations of this system, yields $\sum_i \sum_j a_{ij} p_j + \sum_i z_i = \sum_i p_i$. On the other hand, as only one unit of each good is produced, multiplying each equation of (6.a) by the corresponding price and summing up the resulting n equations yields $\sum_j \sum_i a_{ij} p_j + \sum_j c_j p_j = \sum_j p_j$. Now, substituting the right side of the last equation for the left side of the previous one we obtain $\sum_j \sum_i a_{ij} p_j + \sum_j c_j p_j = \sum_i \sum_j a_{ij} p_j + \sum_i z_i$. Simplifying, this results in the first of the following equations.

$$\text{a)} \sum_j c_j p_j = \sum_i z_i \qquad \text{b)} \sum_j c_j p_j = 1 \tag{17}$$

The second equation is satisfied when prices are measured using the value of the surplus. In this case, the two preceding equations imply that for each i, z_i is the fraction of national income equivalent to the sum of wages and profits of branch *i*.

5.02 SECOND CLASS AND SUBSISTENCE ECONOMIES

Multiplying both sides of equation (17.b) by z_i yields:

$$\sum_j z_i c_j p_j = z_i \tag{18}$$

This equivalence permits us to substitute z_i in each equation in (6.b) for the left side of the preceding equation. We obtain:

$$\sum_j a_{ij} p_j + \sum_j z_i c_j p_j = p_i \qquad i = 1, 2, \dots, n$$

Simplifying the left side of each equation yields:

$$\sum_j (a_{ij} + z_i c_j) p_j = p_i \qquad i = 1, 2, \dots, n \tag{19}$$

Let us assume, for each couple (*i*, *j*), the first of the following equations.

$$\text{a)} * a_{ij} = a_{ij} + z_i c_j \quad \text{b)} \sum_j * a_{ij} p_j = p_i \qquad i = 1, 2, \dots, n \tag{20}$$

Therefore, we can write (19) as the second equation system. Unless indicated otherwise, we will refer indistinctly to system (19) and to the corresponding system (20.b). It is important to notice that (19) is a viable economy of the first class. On the one hand, the input matrix *A = $[*a_{ij}]$ is indecomposable. Indeed, each good appearing in the surplus of *A* figures on each line of *A among the inputs and, according to (9), each good employed as a means of production in *A* produces at least one good in the surplus. Hence, in *A each good is basic. On the other hand, each column sum of *A equals one which, according to Theorem 4.C.11 by (Takayama, 1985, 388) implies that the Frobenius root of *A is equal to one.

Due to the procedure followed in the construction of (20.b), a price system satisfying (6.b) also satisfies (20.b). Moreover, according to Theorem 4.B.1 by (Takayama, 1985, 372) there is only one strictly positive solution to (20.b), determined up to a scalar factor. Therefore, if *A* is a viable economy of the second class, for each $z > 0$, there is a subsistence economy *A, such that *A* and *A determine the same system of relative prices.

The preceding analysis sustains the following result.

Proposition 5. *For any given distribution of income* $z > 0$*, every viable economy of the second class can be represented as a viable economy of the first class under the particular form of a subsistence economy.*

Therefore, condition (7) indicates two properties pertaining to a viable economy of the second class when it is represented as a subsistence economy.

5.03 CONDITION (7) AND SECOND CLASS ECONOMIES

In Section 4.01, it was shown that, given a viable economy of the second class, the Leontief's closed model that results following the procedure detailed there satisfies conditions (9), (12), and (13). Besides, in Section 5.02, it is possible to notice that, given a viable economy of the second class, the subsistence economy obtained through the corresponding procedure has the peculiar trait of containing at least one good that participates directly in the production of every good. Or, not every square matrix $A \geq 0$ satisfying (7) complies with these conditions. As a consequence, not all these matrices con be interpreted, with a basis on the procedures indicated, as a viable economy of the first class that represents a viable economy of the second class. This makes it possible to ask the question about the possibility of interpreting every square matrix $A \geq 0$ satisfying (7) as a representation of a viable economy of the second class.

The answer is affirmative on the condition of interpreting every technical coefficient a_{ij} as the sum of the amount of *j* used directly in the production of *i* plus the amount of *j* consumed by the workers of branch *i*. Under this condition, *A* may represent several viable economies of the second class. In order to prove this, we now proceed to build one example.

Let *p* be the solution to system (6.b) which corresponds to a given matrix *A* and let δ be any number such that $0 < \delta < 1$. For every pair (*i, j*) we define the technical coefficients $*a_{ij} = \delta a_{ij}$, $g_{ij} = (1 - \delta)\, a_{ij}$, $*l_i = \sum_j g_{ij} p_j$ and $*c_j = \sum_i g_{ij}$. It is worth noting that the Frobenius root of matrix $*A = [*a_{ij}]$ is smaller than one, and also that the $*l_i$ satisfy (4.b), and furthermore, that, if prices are measured using the sum $\sum_j * c_j p_j$, then (16.b) is true. As a result, the production program that uses the means of production defined by $*A$, and, in each branch *i*, $*l_i$ units of labor, is a viable economy of the second class in its technical form. In this economy, when the whole national income goes to wages, in each industry *i* the workers can buy the collection of goods formed by g_{ij} units of good *j*, for each *j*. Assuming they do that, matrix *A* indicates, for each pair (*i, j*) the sum of the amount of *j* used as means of production on industry *i* plus the amount of the same good consumed by the workers of that industry.

6.0 THREE ALTERNATIVE APPROACHES TO VIABLE ECONOMIES

Given that the expression viable economy has been employed by other authors in reference to certain types of linear production systems, we will consider briefly three alternative approaches of this concept in order to distinguish each one of them from the one introduced in this paper.

6.01 JUST VIABLE ECONOMIES

(Kurz and Salvadori, 1995, 60, 96) present the following definition.

Definition 3. *An economy A is viable if a vector x exists such that:*

$$\text{a) } x^T \geq x^T A \qquad \text{b) } x \geq 0 \qquad \text{c) } x \neq 0$$

If " ≥" may be substituted for "=" in a)*, then the economy is just viable.*

The economic interpretation, according to these authors, is that viable economies are those that are able to reproduce themselves. On this regard, comparing definitions 1 and 3 we must point out that the first one includes only the economies that are able to reproduce themselves completely while the second one also includes those that are able to reproduce themselves partially. As an illustration, consider the following matrix *A*:

$$\begin{bmatrix} 1/3 & 0 \\ 1/3 & 4/3 \end{bmatrix}$$

In this case, a semi positive vector x satisfies Definition 3 if and only if its second coordinate is zero, which means that the economy cannot produce the inputs required in all the industries.

6.02 STRICTLY VIABLE ECONOMIES

(Bidard, 2004, 13, 32), adopts the next definition.

Definition 4. *An economy A is viable if a vector x exists such that either the first three or the last three of the following conditions are verified:*

a) $x^T > x^T A$ b) $x \geq 0$ c) $x \neq 0$ d) $x^T = x^T A$

The economy is strictly viable in the first case and just viable in the second one.

As occurs in Definition 3, this one also includes certain economies that are not able to reproduce themselves completely. For instance, the following matrix A:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

In this case, a semi positive vector x satisfies Definition 4 if and only if its second coordinate is zero, which means that the economy cannot produce the inputs required in all the industries. However, these two definitions are not equivalent because, contrarily to the former, the latter includes this type of economies only in the case of just viable economies. Indeed, a) of Definition 4 implies that the strict inequality is satisfied in b). In turn, this last remark implies that the next conclusion is true.

Lemma 1. *The following propositions are equivalent:*

(i) *A is a strictly viable economy.*
(ii) *A is a viable economy of the second class, according to Definition 1.*

This result can be easily checked on the basis of the previous developments. However, in order to illustrate a property of viable economies indicated by (9), we will show that (ii) $\Rightarrow$ (i) when A is an economy of the second class where some goods are not produced in surplus. To this end, we will extend to such an economy, which is not necessarily basic,[10] an argument presented by (Bidard, 2004, 31) and (Fisher, 1965, 447) in relation to basic economies.

If (ii) is true, it is possible to reduce the quantity produced of each good j appearing in the surplus in the corresponding $c_j/2$ units. In the new production program, no means of production is consumed in a larger quantity than before. Thus, the new net product still contains those goods that were previously there. In addition, because they will be consumed in a smaller quantity than before, it will also contain those goods that were not contained in but produced directly the net product. If there are some remaining goods not produced in surplus, the procedure may be repeated (for each j, c_j indicates the quantity of j in the most recent net product) until all the goods used either directly or indirectly to produce the original net product are obtained in surplus. It follows from (9) that at this point every good is produced in surplus.

We must add that in the case of economies of the first class, contrarily to Definition 1, both Definition 3 and Definition 4 include non-basic economies, as illustrated by (10.a). The peculiarities distinguishing

[10] An economy is said to be basic if all the goods are basic.

definitions 1, 3 and 4 from one another are due, from our point view, to the fact that each author chooses, from among the several possible interpretations of a concept, the one that fits best his particular research project.

6.03 TWO VIABILITY CONDITIONS

(Pasinetti, 1977, 63, 78) proposes the following two viability conditions:

$$\text{a)} \quad \lambda_A < 1 \qquad\qquad \text{b)} \; \lambda_A \leq 1 \qquad (21)$$

Unfortunately, Section 5.4 of Chapter V, where he presents condition (21.b), does not permit to formulate a definition of viable economies without contradicting at least part of its content. Indeed, the section contains the following propositions: a) (21.a) and (21.b) are the same condition, b) when the wage equals zero, the rate of profit must be non-negative, and c) if (21.b) were not satisfied "we should be dealing with an economic system so technically backward that it could not generate a profit even with a zero wage rate". The first proposition is not correct because (21.a) excludes subsistence economies while (21.b) includes them. The second proposition implies that viable systems must comply with (21.b) but the third proposition attributes to economies satisfying (21.b) a property that not all of them possess. In fact, that property requires of an economy to comply with (21.a).

7.0 FINAL REMARKS

As already noted, the different definitions of viable economies conform to the various purposes of the research programs acting on the field of production theory. Ours differs from the others considered here in that it includes only those economies that have the possibility of reproducing all the production means they consume. It is also important to underscore the fact that our definition contributes to highlight the elements common to Leontief's and Sraffa's theories studied in this paper, particularly the fact that each of the models mentioned can be understood as a special representation of a viable economy of the second class, which in turn, as demonstrated, has the consequence that all goods must be considered as basic.

The last result shows the circular character of economic reproduction under the particular approach of each model. The differences involved, already indicated in the introduction, result in a different relation between, on the one hand, a second class viable economy and, on the other hand, the corresponding Leontief's closed model and Sraffa's subsistence economy. Indeed, under the procedures adopted, there is only one Leontief's closed model for any given second class economy and vice versa. But, although there is only one Sraffa's subsistence economy for any given second class viable economy, starting with any given viable economy of the first class there are several viable economies of the second class that may be built and such that, under the assumptions indicated in Section 5.03 may be represented by the original first class economy.

It can be pointed out also that the results mentioned contribute to justify, from the economic perspective, the mathematical conditions required by viable economies of the first class. Finally, we indicate the original results, as far as we know, which are the most important: Definition 1 and Propositions 2 to 5.

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APPENDIX

Proof of Theorem 1

(I). (i) and (ii) ⇒ (7). Assuming that (i) and (ii) are true but not (7.a) leads to a contradiction. Indeed, in this case *A* may be written in the form:

$$\begin{bmatrix} C & 0 \\ D & E \end{bmatrix}$$

where *C* and *E* are two non-empty square matrices. Proposition (i) implies that every row and every column of *A* contains at least one coefficient greater than zero and, for this reason, $E \neq 0$. Let D_C and D_E be the sets of indexes corresponding to the rows of *C* and *E*, respectively. Then, the equation $Ep^E = \lambda_E p^E$, where λ_E is the Frobenius root of *E*, has a solution $p^E \geq 0$, $p^E \neq 0$, according to Theorem 4.B.2 by (Takayama, 1985, 375). Consequently, the vector *p* in which $p_j = p_j^E$ if $j \in D_E$ and $p_j = 0$ if $j \notin D_E$ satisfies (6.b), contradicting (ii). On the other hand, if (7.a) is true, it follows from (i) of Theorem 1 and (iv) of Theorem 4.B.1 by (Takayama, 1985, 372) that (7.b) is also true.

(II). (7) ⇒ (i) and (ii). If (7.b) is true, we may write equations (6.a) and (6.b) respectively in the form $A^T x = \lambda_A x$ and $Ap = \lambda_A p$. According to Theorem 4.B.1 by (Takayama, 1985, 372), these equations and (7.a) imply (i) and (ii).

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