Wages Paid in Kind in Self-Replacing Economies

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ABSTRACT

This paper studies the two systems of production equations corresponding respectively to wages paid entirely or partially in kind in Sraffa's self-replacing economies. Regarding the first system, the paper introduces the concept of ρ-shaped matrix, which is relevant to the subject and allows also for the addition of a complementary remark to the theory of semi-positive square matrices. In relation to the second system, it points out some differences between the cases studied here and those where the whole wage is paid in value.

1. Introduction

The production of commodities by means of commodities (*PCMC*), studied by Sraffa (1960) within the framework of economies in a self-replacing state (*ESRSs*), synthesizes some macroeconomic aspects of production and income distribution. It focuses on the interdependency of prices through the production relations between the corresponding goods and labor, represented by means of a linear equation system expressing the price of each good as a function of the price of every good or, alternatively, as a function of these prices and of wages.

For the purposes of our research, it is important to highlight one peculiarity presented by labor compared to goods in Sraffa's theory. Although all inputs are assumed to be introduced simultaneously in the production process, goods are paid upon delivery, whereas labor may also be paid afterwards partially or totally. Due to this fact, the influence of labor in price formation depends on the schedule for payment of wages and, as we will see, it also depends on whether the payment is made in value, in kind or in a combination of both. In order to identify in a simple manner the different possibilities, we use notations *S1* to*S4* to designate in each case a particular form of wage payments as well as the corresponding model or equation system determining prices. *S1* consists of a single payment made in kind starting production. *S2* comprises two parts, a payment made in kind on starting production and a fraction of the value of the net product paid when production is completed. In *S3* and *S4*, this fraction is the whole payment and is paid respectively at the end and at the start of production.

In this paper, we investigate price formation in the case of single production (each industry produces only one particular good) when wages are paid entirely or partially in kind. The paper is divided in six sections, including this introduction, whose main contents we will now indicate succinctly. In Section 2, we present a brief survey of the related literature and a remark on the methodology employed. In Section 3, we define the methods of production and other related concepts. In Section 4, two cases of *S1* distinguished by Sraffa are considered, the subsistence economy (*S1-A*) and the economy with a surplus (*S1-B*). We introduce the concept of ρ-shaped matrix and prove a result concerning ρ-shaped and indecomposable matrices adding a complementary remark to the theory of semi-positive square matrices. It follows from this result that, given a viable method of production, there is a unique solution to *S1* comporting a semi-positive price system and a non-negative profit rate if and only if the coefficients matrix is either indecomposable or ρ-shaped. Moreover, in the solution, the price system is positive and unique up to a scalar multiple. On the other hand, as this condition is always satisfied in *S1-A* but may or may not be satisfied in *S1-B*, depending on the wages paid in each industry, we identify a particular wage-goods allocation problem (*WGAP*) in *S1*. In Section 5, referred to *S2*, we show that if the method of production is viable, relative prices are the same as in *S3* but not prices in wage units, which may be decreasing functions of the profit rate. Also, we show that wages may

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be greater than zero when the rate of profit reaches its maximum value and we identify a particular *WGAP* in *S2*. In the last section, we present some comments of a general character.

2. Related Literature and Methodology

In this Section, in order to circumscribe the place of our research in the economic literature, we indicate some of the main landmarks among the numerous publications dealing with linear production models that have special interest for the theory of prices of production. We also present a remark about the methodology employed.

2.1 A Brief Theoretical Review

The relation between prices, production and income distribution is one of the topics that has been studied and discussed through the history of economic thought, at least since Quesnay proposed a scheme of the reproduction of the economy as a whole (Kuczynski & Meek, 1972). One of the first important results in this field is due to Smith (1981), who establishes the equality between the price of each good and the total sum of revenue from the production of that good, which was criticized by Marx (1991). For his part, Marx contributed to enrich the instruments of analysis employed in these studies with his views on economic reproduction. Marx also introduced the concept of price of production, which is still used nowadays, to designate, simply put, the sum of the price of the means of production and labor consumed producing a good plus the profit determined by the average profit rate and the corresponding investment. Dmitriev (1974) built a system of linear equations to determine prices which supports Smith's views in the discussion mentioned. His system also constitutes an important precedent to the model built by Leontief (1960), which determines prices given the distribution of revenue among the different branches of industry. Gale (1960) and Ten Raa (2005) offer two important surveys of the developments and applications of Leontief's model.

As already indicated in the previous section, Sraffa (1960) constructs the *S1*, *S2* and *S3* models. The study of the last model occupies most of his book and it is also the focus of more recent research dealing with prices of production (e.g., Bidard, 2004; Kurz & Salvadori, 1995; Pasinetti, 1977). In these works, extensive surveys can be found of the literature on prices of production and also some observations on the *S4* model. Except for this case, works dealing with some form of wage payment different from *S3* are rare although our author attributes some importance to them, in particular to*S2*. Indeed, Sraffa (1960, 10) affirms that, with this form, it is possible to establish the same results valid for *S3* and also that it is the most appropriate form to treat wages.[3]

2.2 A Remark on the Methodology Employed

Linear models of single-product industries with no fixed capital similar to the one studied here have been discussed by several authors (e.g., Broome, 1983; Dorfman, Samuelson & Solow, 1958; Hawkins, 1948; Leontief, 1986; Morishima, 1973; Roemer, 1983). They allow for the determination of a unique price system corresponding to each given level of wages or, equivalently, of the rate of profit. Because, as a general rule, these models assume the demand as a given, they offer the possibility of studying in a simple manner different aspects of the economy. In this respect, it is important to underscore that we consider in this paper only certain aspects of models *S1* and *S2* involving no changes in the production program, which we take as given.

3. The Methods of Production

Sraffa studies an economy integrated by n ($n \geq 1$) industries, each one producing a particular type of good labeled i or j so that $i, j = 1,2,\ldots,n$. A single unit of each good is produced and there are two distinctive dates: in the first one, goods and labors are introduced in the economic activities and, in the second one, the goods are obtained in every industry.

[3] "In any case the discussion which follows [referred to *S3*] can easily be adapted to the more appropriate, if unconventional, interpretation of the wage suggested above [*S2*]."

We will refer to a set $\{ j1, j2,..., jd,..., jD \}$ as a D-set if it contains D different goods.[4] On the other hand, for each pair (i,j), we define the non-negative technical coefficients b_{ij}, d_{ij} and $a_{ij} = b_{ij} + d_{ij}$ representing respectively the quantity of j used as means of production, paid as wage in kind, and consumed in branch i during the period considered. A good j produces a good i (not necessarily different) either directly if $a_{ij} > 0$ or indirectly if there is a D-set containing neither i nor j and verifying $a_{i,j1} a_{j1,j2}\, a_{j2,j3} \dots\, a_{jD,j} > 0$. In both cases we say that j produces i or, equivalently, that j is connected to i. The technical coefficients of an economy will be represented by matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $D = [d_{ij}]$ and, for each i, $A_i = [a_{i1}\, a_{i2}\, \dots a_{in}]$. These matrices are related by the first of the following propositions:[5]

$$\text{a)} \quad A = B + D \qquad \text{b)} \quad A_i \neq 0 \tag{1}$$

We assume that each industry consumes at least one good, which implies that the second proposition is valid for every i.

Definition 1. *The real income and the net product of an economy are vectors* $\mathrm{c} = (\mathrm{c}_1, \mathrm{c}_2, \dots, \mathrm{c}_n)$ *and* $c = (c_1, c_2, \dots, c_n)$*, respectively, where, for each j:*

$$\text{a) } \mathrm{c}_j = 1 - \sum_i b_{ij} \qquad \text{b) } c_j = 1 - \sum_i a_{ij} \tag{2}$$

Wages and prices are measured with the value of the real income; the part of it corresponding to the labor used in the different industries is w. For each i, l_i and p_i are respectively the fraction of the value part of the wage that corresponds to workers in industry i and the price of good i. These fractions are represented by the n x 1 matrix $L = [l_i]$ and the price system by the column vector $p = (p_1, p_2, \dots, p_n)^T$.

Definition 2. *For each i, the method of production consists in the quantity of each good consumed, together with the fraction of the wage paid in value in industry i.*[6]

The methods employed in the economy are represented by matrix $M = [A\ L]$. Nevertheless, in this paper, our attention will be focused on matrix A. For this reason, in order to simplify, we will refer to A as an economy.

Definition 3. *A class is a* D*-set that may be of two different types. In the first type: a) each good in* D *produces all the goods in* D*, and b) if j produces* D *and* D *in turn produces j then j belongs to* D*. In the second type,* D *contains a single good that does not produce itself.*

Given an economy A, let F be the number of classes and f an index assigned to classes in such a way that the succession:

$$\mathrm{F} = \mathrm{D}_1, \mathrm{D}_2, \dots, \mathrm{D}_f, \dots, \mathrm{D}_F \tag{3}$$

complies with the following conditions: a) no class produces any class at its left, and b) the last class of the first type (in case there is one), from left to right, produces every class at its right. These conditions may be satisfied because, given any pair of different classes, at most one of them can produce the other. On the other hand, if there are two classes such that neither one produces the other, F is not unique. However, for each f, classes at the left of D_f are not produced by D_f and those to its right do not produce it. As shown in the Appendix A.1, F is related to the canonical form of matrix A.

Lemma 1. *The set* D_1 *in F is a class of the first type.*

Proof. Let us assume that D_1 is a class of the second type. From Definition 3 and (1.b) it follows that at least one good not belonging to D_1 produces D_1. Hence, this good belongs to a class at the left of D_1, something that is not possible.

[4] We will refer to indexes also as goods.

[5] Given two matrices (A, B) or two vectors (x, y), the relations $A = B$ and $x = y$ means respectively that $a_{ij} = b_{ij}$ for every couple (i,j) and $x_j = y_j$ for every j. We define each one of the relations ">", "<", "≥" and "≤" in a similar manner while the relation "≠" means that "=" is not true. If all the entries of a matrix or a vector are equal to zero we may represent it with 0.

[6] The definition is based on Sraffa (1960, 3, 43).

Definition 4. *Let A be a square matrix such that* $A \geq 0$. *A is viable if it satisfies, eventually after changing the quantities produced, the following conditions, of which the last one may not be satisfied by the set of all goods.*

(i) $A_i \neq 0$ *for every i.*

(ii) *In every* D*-set* $\sum_d a_{id,j} \leq 1$ *for every* $j \in$ D.

(iii) *In every* D*-set* $\sum_d a_{id,j} < 1$ *for at least one* $j \in$ D.

A viable economy A is a subsistence economy if (iii) of Definition 4 is not satisfied by the set of all goods, otherwise A is an economy with a surplus. This definition covers the *ESRSs* as well as those economies able to be in such state after changing the quantities produced and the units of measure employed. In the following sections, we will only consider *ESRSs*. On the other hand, we will say that a means of production matrix B is viable if the matrix A resulting according to (1.a) is viable when $D = 0$.

It is important to remark that, in subsistence economies, A is indecomposable. Indeed, given any index, let D be the set of all the indexes to which that index is connected. If $D < n$, (iii) of Definition 4 implies that $a_{ij} > 0$ for at least one couple (i,j) such that $j \in$ D and $i \notin$ D, contradicting D's definition. Thus, $D = n$.

4. Wages Paid Entirely in Kind

We represent by r the rate of profits, assumed to be the same in every industry. If wages are paid entirely in kind at the first date, the production equations form the following system.

$$\sum_j a_{ij}\, p_j (1 + r) = p_i \qquad i = 1,2, \dots, n \tag{4}$$

Let λ be a scalar (real or complex). It is possible to represent (4) by means of the first of the following equations, while the second one relates r and λ when $\lambda \neq 0$.

$$\text{a) } Ap = \lambda p \qquad \text{b) } r = (1 - \lambda)/\lambda \tag{5}$$

The Frobenius root of A will be represented with λ_A. The next definition refers to the canonical form of A presented in the Appendix A.1.

Definition 5. *Let A be a decomposable square matrix such that* $A \geq 0$, $A \neq 0$. *A is* ρ*-shaped if there is only one indecomposable matrix in the main diagonal of its canonical form and, for each couple (i,j) such that* $j \in$ D *(the set of indexes of the indecomposable matrix) and* $i \notin$ D*, j is connected to i.*[7]

Accordingly, a matrix A is ρ-shaped if the index set D may be partitioned in two non- empty subsets D_1 and D_2 such that each index $j \in D_1$ is connected to every index in D while each index $j \in D_2$ is not connected to itself and may or may not be connected to other indexes. Therefore, if A is an input matrix, A is ρ-shaped if only basic goods produce themselves. In addition, there must be at least one basic and one non-basic good in the economy.

The following proposition relates viable economies and ρ-shaped matrices.

Lemma 2. *If there is only one class of the first type in* F *and* $F > 1$*, goods belonging to the first class are basic and A is* ρ*-shaped.*

Proof. Let i be the first class of the second type, from left to right, in F. Neither i nor a good j belonging to a class at the right of i produces i. Hence, as (1.b) is true, D_1 produces i. If i is the second class of the second type in the same order, the preceding argument implies that a good j belonging to a class placed at the left of i produces i, which implies that D_1 produces i. The proof is completed following successively a similar procedure.

[7] Indecomposable matrices are also called irreducible.

Therefore, if an input matrix A is indecomposable or ρ-shaped there is at least one basic good. However, if there is at least one basic good, A is not necessarily indecomposable neither ρ-shaped because the economy may possess more than one class of the first type. The following proposition presents some important aspects of the non-negative solutions of (5.a) which are well-known, with the exception of (iii), (iv) and (v).

Theorem 1. *Let A be a square matrix such that* $A \geq 0$ *and* $A \neq 0$. *Then, in equation (5.a)*:

(i) *There is at least one solution* $(\lambda, p) \geq 0$ *in which* $p \neq 0$.

(ii) *If A is indecomposable there is only one solution. Furthermore,* $(\lambda, p) > 0$, $\lambda = \lambda_A$, *and p is unique up to a scalar multiple.*

(iii) *If A is* ρ-*shaped there is only one solution such that* $(\lambda, p) > 0$. *Furthermore,* $\lambda = \lambda_A$ *and p is unique up to a scalar multiple.*

(iv) *If A is* ρ-*shaped and* $(\lambda, p) \geq 0$ *is a solution in which* $p \neq 0$ *and* $p_j = 0$ *for at least one index j, then* $\lambda = 0$.

(v) *If at least one index is connected to itself but A is neither indecomposable nor* ρ-*shaped there is at least one solution with* $\lambda > 0$ *and* $p_j = 0$ for *at least one j.*

(vi) *If no index is connected to itself in every solution,* $\lambda = 0$.

Proof. See Appendix A.2.

This theorem permits to add a complementary observation to the theory of semi-positive square matrices not previously published, as far as we know (e. g., Berman and Plemmons, 1994; Takayama, 1985). Indeed, regarding (5.a), the Frobenius' Theorem for semi-positive decomposable matrices guarantees the existence of at least one solution $(\lambda, p) \geq 0$ in which $p \neq 0$ whereas Theorem 1 allows for the following conclusion:

Corollary 1 to Theorem 1. *Let A be a square matrix such that* $A \geq 0$ *and* $A \neq 0$. *Then, if and only if A is either indecomposable or ρ-shaped, in equation (5.a):*

(i) *There is a solution* $(\lambda^*, p^*) > 0$, *in which* $\lambda^* = \lambda_A$ *and* p^* *is unique up to a scalar multiple.*

(ii) *There is no other solution* (λ, p) *in which* $\lambda > 0$, $p \geq 0$ *and* $p \neq 0$.

Given (5.b), these results may be stated in the following terms:

Corollary 2 to Theorem 1. *Let A be a square matrix such that* $A \geq 0$ *and* $A \neq 0$. *If and only if A is either indecomposable or* ρ-*shaped, in system (4):*

(i) *There is a solution* (r^*, p^*), *in which* $r^* \in \,]-1,\infty[$, $p^* > 0$ *and* p^* *is unique up to a scalar multiple.*

(ii) *There is no other solution* (r, p) *in which* $r \in \,]-1,\infty[$, $p \geq 0$ *and* $p \neq 0$.

This corollary is relevant to model *S1* because it establishes a necessary and sufficient condition for the existence of a meaningful solution (r^*, p^*) to (4) not satisfied only by economies of minor or no interest to Sraffa's theory. Indeed, the existence of several solutions (r, p) to (4) in which $r \in \,]-1,\infty[$, $p \geq 0$ and $p \neq 0$ indicates that the production system does not determine prices. On the other hand, a solution to (4) in which at least one price is zero means that the corresponding industrial branch produces a good given in exchange for nothing. Furthermore, $r = \infty$ refers to the limit case when either something is produced with no investment or an infinite quantity of a good is produced with a finite investment. Finally, we have $r \leq -1$ when at least the whole investment is lost. However, Sraffa considers only *ESRSs*, of which one well-known property is that $r \geq 0$. The following proposition characterizes the solution to (4) for these economies.

Theorem 2. *If A is viable, there is a solution* $(r, p) \geq 0$ *to* (4) *such that:*

(i) *If A is a subsistence economy,* $r = 0$ *and if A produces a surplus,* $r > 0$.

(ii) *If there is only one class of the first type in the corresponding succession* Ϝ, *the solution is unique. Moreover,* $p > 0$ *and p is unique up to a scalar multiple.*

(iii) *If there is more than one class of the first type in* Ϝ *there is at least one solution with* $p \neq 0$ *and* $p_j = 0$ *for at least one index j.*

Proof. See Appendix A.3.

Therefore, if A is viable, a necessary and sufficient condition for a solution $(r^*, p^*) \geq 0$ to exist for system (4) in which $p^* > 0$ with no other solution $(r, p) \geq 0$ in which $p \neq 0$, except if p is a multiple of p^* and $r = r^*$, is that there is only one class of the first type in the corresponding succession F or, equivalently, that matrix A is either indecomposable or ρ-shaped. If this is not the case, (4) may have indeed a solution $(r^*, p^*) \geq 0$ with $p^* > 0$, but then there is also another solution $(r, p) \geq 0$ with $p_j = 0$ for at least one index j, as illustrated by the example (6.c) ahead.

As shown in Section 3, in viable subsistence economies the corresponding matrix A is indecomposable. Thus, the aforementioned condition is always satisfied in model *S1-A*. On the other hand, model *S1-B* may or may not satisfy this condition. Moreover, as (4) can represent a great variety of production relations among goods, there are many possible cases regarding the number of solutions and their properties. As an illustration, it is useful to consider the following examples.

$$\text{a)} \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \qquad \text{c)} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \tag{6}$$

In each one of the three cases we have a viable economy producing a surplus. In (6.a) and (6.b), there is only one class of the first type, and there is also a class of the second type in (6.b), while in (6.c) there are two classes of the first type in the economy. Therefore, according to Theorem 2, there is only one solution $(r, p) \geq 0$ with $p \neq 0$ to (4) in the first two cases and, in the third one, there is at least one non-negative solution including at least one price equal to zero. The solutions are: (6.a) $r = 1$, $p = (1,1)$; (6.b) $r = 1$, $p = (1,1)$, in this case (5.a) has also the solution $\lambda = 0$ and $p = (0,1)$; (6.c) $r = 1$ with $p = (1,1)$, $p = (1,0)$ or $p = (0,1)$. Other solutions are obtained in each case multiplying the price system by a positive scalar. Sraffa (1960, 8) assumes that in each viable economy A there is at least one basic good, diminishing in this manner the diversity mentioned above. However, as already indicated, this restriction is not sufficient to guarantee the existence of a single positive solution.

The previous analysis show that, under model *S1*, not all viable methods of production determine a solution to (4) in which the system of relative prices is unique, each price is positive and the profit rate is non-negative. Indeed, the necessary and sufficient condition just established excludes a large set of viable economies. On this regard, it is important to remark that, according to (1.a), given a viable means of production matrix B the properties of A will be largely determined by the quantities of goods paid in kind in each industry, defined by the wage goods matrix D. For instance, if B is equal to (6.c), the resulting matrix A will be indecomposable if wages paid in the first and the second industry include the second and the first good, respectively. Otherwise, A will not be indecomposable neither ρ-shaped. Hence, the following question is relevant to model *S1*.

Wage Goods Assignment Problem (*WGAP*) in *S1*. *Given a viable matrix B, what conditions are required on matrix D in order for the resulting matrix A to be both viable and either indecomposable or* ρ*-shaped?*

We do not offer a general answer to this question. However, Benítez & Benítez (2014) establishes a condition on D that is sufficient for A to be both viable and indecomposable.

5. Wages Paid Partially in Kind and Partially in Value

Let ω be the fraction of the value of the net product paid to the workers in model *S2*. The production equations are as follows:

$$\sum_j a_{ij}\, p_j (1 + r) + l_i \omega = p_i \qquad i = 1,2, \dots, n \tag{7}$$

The unit of measure chosen implies the following equation:

$$\sum_j c_j\, p_j = 1 \tag{8}$$

The real income and the net product are equal only when $d_{ij} = 0 \; \forall \, (i, j)$, otherwise the first one includes two parts, the net product and the constant part of the wage. In this case, the wage as a fraction of the value of the real income is determined by the equation:

$$w = (\omega + \sum_i \sum_j d_{ij} \, p_j)/(1 + \sum_i \sum_j d_{ij} \, p_j) \qquad (9)$$

Sraffa's quote in Note 3 suggests that the conclusions reached in his work regarding prices and income distribution in model *S3* may also be valid in model *S2* after certain adaptations are made. As he did not give more precisions about the conclusions involved nor about the adaptation procedure, it is important to indicate that certain results valid in *S3* are not true in *S2*, as shown next.

Proposition 1. *When the rate of profit reaches its maximum value (R),* $w = 0$ *in S3 but it is possible that* $w > 0$ *in S2.*

Proof. According to Appendix A.1 in Benítez & Benítez (2014), in *S3*, $w = 0$ when $r = R$. Therefore, if A is viable, when $r = R$, in *S2* we have $\omega = 0$ and the right side of (9) is equal to $\sum_i \sum_j d_{ij} \, p_j / (1 + \sum_i \sum_j d_{ij} \, p_j)$. Hence, $w = 0$ if $p_j = 0$ for each j integrating the constant part of the wage. But if at least one of these prices is greater than zero, then $w > 0$. This occurs, for instance, when all goods are basic.

Proposition 2. *Prices measured in wage units are increasing functions of r in S3 but they may be decreasing functions of r in S2.*

Proof. Regarding prices in wage units in *S3,* see Theorem A.1 by Benítez & Benítez (2014). To complete the proof, let us consider the following system of type (7).

$$[(1/3)p_1 + \; a_{21}p_2](1 + r) + 1/2\omega = p_1$$
$$[(1/6)p_1 + 1/2p_2](1 + r) + 1/2\omega = p_2$$

First, we will assume that $a_{21} = 0$. In this case, net product and capital are both equal to $(1/2)p_1 + (1/2)p_2$, so that $R = 1$. Making $\omega = 1$ and solving for p_1 in the first equation and for p_2 in the second one, we obtain each price measured with the part of the wage unit paid in value:

$$p_1 = (1/2)/[(2/3) - r/3]$$
$$p_2 = [(1/2) + (1/6)p_1(1 + r)]/[(1/2) - r/2]$$

From these equations, it follows that $p_1 = 3/4$ and $p_2 = 5/4$ when $r = 0$. Moreover, when R tends to *1*, p_1 tends to 3/2 and p_2 tends to infinitum, so that p_1/p_2 tends to zero. Let us suppose that the constant part of the wage consists in 1/2 of Good 1 consumed in the first industry and 1/4 of Good 2 consumed in the second one. Therefore, the price of the first good in wage units is equal to $p_1/[1 + (1/2)p_1 + (1/4)p_2]$, changing from 12/15 when $r = 0$ to zero when $R = 1$. It is worth adding that, if all goods considered are to be basic, we may assume that a_{12} is as small as necessary for that price to change between quantities as close to 12/15 and, on the other extreme, as close to 0 as decided.

On the other hand, comparing (7) presented above and (A.2) in Benítez & Benítez (2014), we may remark that prices determined by both systems are the same when $t = 0$. Therefore, the propositions in Theorem A.1 (with the possible exceptions of c) and d)) on that paper are also valid for (7) on the condition that the corresponding matrix A is viable.

Contrarily to what is required in model *S1*, in this case A may not be indecomposable neither ρ-shaped. However, as indicated in the last section, given a viable matrix B the viability of A depends on D so that we may formulate the following question regarding wage goods in *S2*.

Wage Goods Assignment Problem (*WGAP*) in *S2*. *Given a viable matrix B, what conditions are required on matrix D in order for the resulting matrix A to be viable?*

The sufficient condition that answers the *WGAP* in *S1* indicated in Section 4 is also a sufficient condition in this case.

6. Concluding Remarks

As noted in Section 2, in contrast to *S3*, models *S1* and *S2*, the objects of this study, have received little attention in the specialized literature. This is probably due in some measure to the fact, already mentioned, that Sraffa himself states that it is possible to derive from *S2* the same results as those derived from *S3*. For this reason, it is important to underscore that, as demonstrated in the preceding pages, there are substantial differences on those results, which points to the need to study more carefully the differences between models *S1* and *S2* on one hand and *S3* on the other hand. This is of interest both to the history of economic thought and also as a way to explore the differences between real and value magnitudes considered by economic science.

Also regarding these topics, it is possible that future researches will find useful Theorem 2, which establishes a necessary and sufficient condition for the existence of a unique solution $(r, p) \geq 0$ for model *S1*. Nevertheless, it is worth mentioning that its main value consists in answering a fundamental question about the *S1* equation system, which as far as we know, has not been provided before.

Finally, it is necessary to say that the mention in our paper of some results already known is required to complete the systematic approach that we chose to follow. Moreover, these results are presented here in a particular formalization that is compatible with our approach and, in general, they are identified as known results in the text. Nevertheless, for further clarification on this matter, we will conclude by pointing out the main contributions contained in each section that, to the best of our knowledge, have not been previously published. Section 3: Definition 3 and Lemma 1; Section 4: Definition 5, Lemma 2, points (iii), (iv) and (v) of Theorem1, corollaries 1 and 2 to Theorem 1, points (ii) and (iii) of Theorem 2; Section 5: propositions 1 and 2.

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APPENDIX

A.1 Canonical form of matrix *A*

In order to study system (4) it is necessary to reassign the indexes as follows. Given a square matrix A such that $A \geq 0$ and $A \neq 0$,the n indexes are placed in the succession:

$$J = j1, j2, \ldots, jn$$

according to the following rules: a) if j is connected to i but i is not connected to j, the first index precedes the second one (from left to right), b) if i is connected to j and j to i, between these two indexes either there are no indexes or there are only indexes k such that i is connected to k and k to j, and c) if there is at least one index connected to itself, let k be the last index (from left to right) presenting this property. Then, either k is the last index in J or k is connected to all indexes to its right on J. Once succession J is established thus, indexes are reassigned giving to each line and column the index corresponding to its position in this succession. Also, columns and lines in A are permuted, ordering them according to the new indexes. Then, matrix A may be presented on the following canonical form:

$$\begin{bmatrix} A_{11} & 0 & 0 \ldots 0 \\ A_{21} & A_{22} & 0 \ldots 0 \\ & \ldots & \\ A_{F1} A_{F2} & & \ldots A_{FF} \end{bmatrix} \qquad (A.1)$$

Here, for each f, A_{ff} is a square matrix either indecomposable or equal to zero. The canonical form of a square matrix is not necessarily unique and usually the reassignment of indexes satisfies only conditions a) and b), for instance in Seneta (1973). We added condition c) because, in this manner, given an input matrix A, in (A.1), for each f, the indexes corresponding to matrix A_{ff} may be (although not necessarily) the same as those in class f in (3). On the other hand, the same condition implies that:

Proposition A.1. If A is ρ-shaped, in the corresponding canonical form matrix A_{11} is indecomposable.

A.2 Proof of Theorem 1

We will consider successively the six parts.

(i) and (ii) See Theorem 4.B.2 by Takayama (1985, 375) and Theorem 4.B.1 by Takayama (1985, 372), respectively.

(iii) Let λ_1 be the Frobenius root of A_{11}, k the number of lines in A_{11} and $p^1 \geq 0$ a column vector with k lines such that $p^1 \neq 0$. According to Proposition A.1 and Theorem 4.B.1 just quoted, the equation $A_{11}p^1 = \lambda p^1$ admits only one solution. Furthermore, $(\lambda, p^1) > 0$, $\lambda = \lambda_1$ and p^1 is determined up to a scalar multiple. Because A is ρ-shaped no index greater than k connects with $k+1$ and at least one $j \leq k$ verifies $a_{k+1,j} > 0$, so that $a_{k+1,j}p_j > 0$. Substituting the first k prices and λ_1 for their corresponding values in the equation:

$$\frac{\left[\sum_j a_{k+1,j}\, p_j\right]}{\lambda_1} = p_{k+1}$$

yields p_{k+1}, which is unique and greater than zero, proceeding successively in a similar manner the remaining prices are obtained. Hence, the resulting vector p is the only positive vector, up to a scalar factor, satisfying (5.a). Because λ_A is equal to the Frobenius root of at least one of the matrices in the main diagonal of (A.1) we have $\lambda_A = \lambda_1$, proving (iii).

(iv) As indicated in the proof of (iii), the fact that A is ρ-shaped implies that the only vector $p^1 \geq 0$ and such that $p^1 \neq 0$ satisfying $A_{11}p^1 = \lambda p^1$ verifies $p^1 > 0$. It follows from this conclusion and from the same proof that if $p^1 \neq 0$ the solution to (5.a) has no zero entries. Therefore, in this case $p^1 = 0$ and consequently, any index j for which $p_j > 0$ is not connected to itself. Assuming that b is the first index satisfying the last inequality, let us consider the equation $\sum_j a_{bj}\, p_j = \lambda p_b$. Because the first $b - 1$ prices and the last $n - b + 1$ coefficients a_{bj} are equal to zero, the sum in the left side is equal to zero, so that $p_b > 0$ only if $\lambda = 0$, proving (iv).

(v) In this case, there is at least one indecomposable matrix in the main diagonal of (A.1) and A may be arranged in the following form:

$$\begin{bmatrix} B & 0 \\ C & E \end{bmatrix} \qquad (A.2)$$

where, due to restriction c) in the reassignment of indexes, E is a square matrix either indecomposable or ρ-shaped, whereas B and C are non-negative matrices. Let D_B and D_E be the sets of indexes corresponding to the lines of B and E in (A.2), respectively. Given the assumption on E, it follows from (ii) and (iii) that there is a solution $p^E > 0$ for the equation $Ep^E = \lambda_E p^E$, where λ_E is the Frobenius root of E and p^E is a vector determined up to a scalar multiple. Consequently, the vector p in which $p_j = p_j^E$ if $j \in D_E$ and $p_j = 0$ if $j \in D_B$ satisfies (5.a), proving (v).

(vi) In this case $A_{ff} = 0\ \forall f$ in (A.1). Assuming that b is the first index for which $p_b > 0$, let us consider the equation $\sum_j a_{bj}\, p_j = \lambda_A p_b$. Because the first $b - 1$ prices and the last $n - b + 1$ coefficients a_{bj} are equal to zero, the sum in the left side is equal to zero, so that $p_b > 0$ only if $\lambda_A = 0$, proving (vi).

A.3 Proof of Theorem 2

We will consider successively the three parts.

(i) First, we will assume that the input matrix A is indecomposable. According to Theorem 4.C.11 by Takayama (1985, 388), if every column sum is equal to one, $\lambda_A = 1$. Moreover, according to the Corollary to this theorem by Takayama (1985, 389) if each column sum is at most equal to one and at least one is smaller than one, $0 < \lambda_A < 1$. The first case occurs in subsistence economies and the second one in economies with a surplus. These results and (5.b) imply (i) when and if A is indecomposable. On the other hand, if A is decomposable, λ_A is equal to the Frobenius root of at least one indecomposable matrix A_{ff} in the main diagonal of (A.1). For this reason, the preceding argument, referred to A_{ff}, permits to prove (i).

(ii) If there is only one class of the first type, A is indecomposable if there are no more classes in F; otherwise A is ρ-shaped, according to Lemma 2. Then (ii) is true, according to Corollary 2 to Theorem 1.

(iii) If there is more than one class of the first type at least one index is connected to itself but A is neither indecomposable nor ρ-shaped. Therefore, (iii) is true according to (v) of Theorem 1.

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