Wages Paid in Value in Self-Replacing Economies

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ABSTRACT

In this paper, we study the relation between income distribution and prices in economies in a self-replacing state, defined by Sraffa, when wages are paid entirely in value. As a result of our analysis, it is possible to build a model that combines some distinctive features of the different forms of payment considered by this author, facilitates the study of wages as an independent variable and offers an original formalization of the production of commodities by means of commodities. Moreover, the model helps to highlight and explain one particular effect on prices due to wages paid in value.

1. Introduction

In this paper, we study the relations between income distribution and prices in economies in a self-replacing state (*ESRSs*), a concept introduced by Sraffa (1960), when wages are paid entirely in value. With this purpose, we build a model that combines some distinctive features of the different forms of payment considered by this author allowing us to point out certain properties of wages paid in value. In order to identify in a simple manner the forms just mentioned, we use notations *S1* to *S6* to designate in each case a particular form of wage payment as well as the corresponding model or equation system for determining prices. Model *S1* consists of a single payment made in kind as production starts while *S2* comprises two parts, a payment made in kind as production starts and a fraction of the value of the net product paid when production is completed. In *S3* and *S4*, this fraction constitutes the whole wage and is paid respectively at the end and at the start of production. In *S5*, which includes *S3* and *S4* as particular cases, wages are paid entirely in value, a part being paid at the beginning and the rest at the end of production. Appendix A.1 presents a version of the model, originally studied by Benítez (2009), adapted to the requirements of this paper whose reading is necessary for a thorough understanding of Section 3. Indeed, we show in the appendix how, given a distribution of the wage between the two payments, it is possible to determine prices and wages with a basis on the values corresponding to these variables when the whole wage is paid in advance, a result that facilitates the study presented here.

The paper is divided in five sections, including this introduction, whose main contents we will now indicate succinctly. In Section 2, the reference economies are described. In Section 3, we study the wage as an independent variable. It is proved that, concerning relative prices as well as prices measured in wage units, *S5* is equivalent to another model (*S6*), where wages are entirely variable, like in *S3*, and totally advanced in kind, like in *S1*. In this manner, *S6* offers an original presentation of the production of commodities by means of commodities(*PCMC*). In Section 4, we point out an effect on prices due to wages paid in value not previously discussed in the specialized literature, as far as we know. In the last section, we present some comments of a general character.

2. Techniques and Viable Economies

Sraffa studies an economy integrated by n ($n \geq 1$) industries, each one producing a particular type of good labeled i or j so that $i, j = 1,2,\ldots,n$. We will refer to a set $\{j1, j2,\ldots, jd,\ldots, jD\}$ as a D-set if it contains D different goods.[3] There are M ($M \geq 1$) different types of labor labeled m or q so that $m, q = 1,2,\ldots,M$. A single unit of each good is produced and the amount of time dedicated to labor in the production system equals one unit.

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[3] We will refer to indexes also as goods.

There are two distinctive dates: in the first one, goods and labors are introduced in the economic activities and, in the second one, the goods are obtained in every industry.

Definition 1. *For each i, the technique associated with industry i consists in the quantity of each good j used as means of production, received as payment in kind and bought by the workers in industry i, together with the quantity of each type of labor m employed.*

All these quantities, to which we refer as technical coefficients, are non-negative; we represent them respectively with b_{ij},d_{ij}, g_{ij} and l_{im}. Furthermore, we define $a_{ij} = b_{ij} + d_{ij}$ and, on the other hand, we assume that $l_{im} > 0$ for at least one m. A good j produces a good i (not necessarily different) either directly if $a_{ij} > 0$ or indirectly if there is a D-set containing neither i nor j and verifying $a_{i,j1}\, a_{j1,j2}\, a_{j2,j3} \dots a_{jD,j} > 0$. In both cases we say that j produces i or, equivalently, that j is connected to i. Sraffa (1960: 8) calls basic a good that produces every good and non-basic any other good.

The technique in the economy is represented by matrix $T = [B\; D\; G\; E]$ where $B = [b_{ij}]$, $D = [d_{ij}]$, $G = [g_{ij}]$ and $E = [l_{im}]$. We also define $A = [a_{ij}]$ and, for each i, $A_i = [a_{i1}\;\; a_{i2}\; \dots\; a_{in}]$. Three of these matrices are related by the first of the following propositions while, as we assume that each industry consumes at least one good, the second proposition is valid for every i:[4]

a) $A = B + D$ b) $A_i \neq 0$ (1)

The total quantity of j invested and consumed during the present period in the production of i is indicated, for each couple (i,j), by the corresponding coefficient of matrix $C = B + D + G$. For this reason, C may be called the *PCMC* matrix. We assume that none of the column sums in the *PCMC* matrix is greater than one.

Definition 2. *The real income and the net product of T are vectors* $\mathrm{c} = (\mathrm{c}_1, \mathrm{c}_2,\dots, \mathrm{c}_n)$ *and* $c = (c_1, c_2,\dots,c_n)$, *respectively, where, for each j:*

$$\text{a) } \mathrm{c}_j = 1 - \sum_i b_{ij} \qquad \text{b) } c_j = 1 - \sum_i a_{ij} \tag{2}$$

Wages and prices are measured with the value of the real income; the part of it corresponding to the labor used in the different industries is w. For each i, l_i and p_i are respectively the fraction of the value part of the wage that corresponds to workers in industry i and the price of good i. The fractions l_i, which are strictly positive, are represented by the n x 1 matrix $L = [l_i]$ and the price system by the column vector $p = (p_1, p_2,\dots, p_n)^T$.

Definition 3. *For each i, the method of production consists in the quantity of each good consumed, together with the fraction of the wage paid in value in industry i.* [5]

The methods employed in the economy are represented by matrix $M = [A\; L]$. Techniques and methods of production are related, on the one hand, by equation (1.a). On the other hand, if relative prices between different types of labor are not unique, there may be more than one distribution among industries of the value part of the wage. In this case, a single technique may determine several methods of production.

Definition 4. *Let A be a square matrix such that* $A \geq 0$. *A is viable if it satisfies (eventually after changing the quantities produced) the following conditions, of which the last one may not be satisfied by the set of all goods.*

(i) $A_i \neq 0$ *for every i.*
(ii) *In every* D-*set* $\sum_d a_{id,j} \leq 1$ *for every* $j \in$ D.
(iii) *In every* D-*set* $\sum_d a_{id,j} < 1$ *for at least one* $j \in$ D.

[4] Given two matrices (A, B) or two vectors (x, y), the relations $A = B$ and $x = y$ means respectively that $a_{ij} = b_{ij}$ for every couple (i,j) and $x_j = y_j$ for every j. We define each one of the relations ">", "<", "≥" and "≤" in a similar manner while the relation "≠" means that "=" is not true. If all the entries of a matrix or a vector are equal to zero we may represent it with 0.
[5] The definition is based on Sraffa (1960: 3, 43).

A viable economy A is a subsistence economy (*S1-A*) if (iii) of Definition 4 is not satisfied by the set of all goods, otherwise A is an economy with a surplus (*S1-B*). This definition covers the *ESRSs* as well as those economies able to be in such state after changing the quantities produced and the units of measure employed. In the following sections, we will only consider *ESRSs* and we will say that a means of production matrix B is viable if the matrix A resulting according to (1.a) is viable when $D = 0$.

It is important to remark that in a viable economy with a surplus every good employed as a means of production produces the net product. Indeed, given a j employed as a means of production, let D be the set of all the goods produced by j. According to (iii) of Definition 4, at least one good $i \in$ D (not necessarily different from j) is not consumed entirely in the production of the goods of D. If this surplus is not part of the net product then it is consumed in the production of a good not belonging to D, contradicting D's definition. Thus, i is part of the net product.

3. The Wage as an Independent Variable

In this Section, we study the wage as an independent variable. To this end, we build a system of production equations where the technical coefficients constitute a single indecomposable matrix, like in *S1-A*, possibly with a surplus, like in*S1-B*. [6]

Assuming that $t = 1$, multiplying by w both sides in (A.1.b), results:[7]

$$\sum_j \mathrm{w} c_j \mathrm{p}_j = \mathrm{w}$$

This equivalence permits us to substitute w in each equation in (A.2) for the left side of the preceding equation. Taking into account (A.1.a), it is possible to write the result of the substitution in the following form:

$$\sum_j b_{ij}\, \mathrm{p}_j (1+r) + l_i \sum_j \mathrm{w}\, c_j \mathrm{p}_j (1+r) = \mathrm{p}_i \qquad i = 1,2,\dots,n$$

Simplifying the left side of each equation yields:

$$\sum_j (b_{ij} + l_i \mathrm{w} c_j) \mathrm{p}_j (1+r) = \mathrm{p}_i \qquad i = 1,2,\dots,n \qquad (3)$$

Assuming, for each couple (i,j), the first of the following equations, it follows from (1.a) that we can write (3) as the second equation system:

$$\text{a) } d_{ij} = l_i \mathrm{w} c_i \qquad \text{b) } \sum_j a_{ij}\, \mathrm{p}_j (1+r) = \mathrm{p}_i \qquad i = 1,2,\dots,n \qquad (4)$$

We will refer equally to (3) or to the corresponding system of type (4.b) as model *S6*. It is important to remark that, due to the procedure followed in the construction of this model, the next proposition is true.

Proposition 1. *For every* $r \in [0,R[$, *when* $t = 1$ *the price system and the wage determined by* (A.1.b) and (A.2) *also satisfy the corresponding system* (3).

According to (1.a) and (4.a) the coefficient a_{ij} indicates, for each couple (i,j), the sum of the quantity of j consumed in the production of i as means of production plus the quantity of the same good virtually received by the workers of industry i at the first date. This occurs effectively when, for each i, workers of industry i receive as payment the bundle of goods defined by the following 1 x n matrix, calculated from (A.2):

$$U_i = [l_i \mathrm{w} c_1 \; l_i \mathrm{w} c_2 \; \dots \; l_i \mathrm{w} c_n] \qquad (5)$$

[6] indecomposable matrices are also called irreducible.
[7] The notations placed between parentheses and starting with A refer to the Appendix.

In this case, the quantities of wage goods paid to workers in the different industries are specified by the n x n matrix U where for each i, line i is equal to the single line of matrix U_i.

Matrix A, resulting from the substitution just presented, describes a particular set of virtual methods of production. Other methods could be obtained based on a matrix D different from U but verifying the following equation for each i:

$$l_i \mathrm{w} = \sum_j d_{ij}\, \mathrm{p}_j \qquad (6)$$

The new methods result after substituting $l_i\mathrm{w}$ in each equation of (A.2) for the corresponding right side of (6). For this reason, it is important to highlight that the methods determined by matrix U allow us to verify the following two propositions.

Lemma 1. *The following propositions are equivalent: a)* (3) *is viable and b)* (A.2) *is viable.*

Proof. I. a) $\Rightarrow$ b). Given a D-set and a $j \in$ D we have:

$$\sum_d (b_{id,j} + l_{id}\mathrm{w}c_j) \leq 1 \qquad (7)$$

Because $\sum_d l_{id}\mathrm{w}c_j \geq 0$, (7) and (A.1.a) imply that the couple (D, j) satisfies (ii) of Definition 4 in (A.2) and, consequently, D is self-sufficient in j in (A.2). The same argument implies that if the couple (D, j) verifies the inequality in (7) it also does it in (iii) of Definition 4. Hence, if D has a surplus of j in (7) it also has it in (A.2). II. b) $\Rightarrow$ a). Given a D-set and a $j \in$ D, equations (2.b) and (A.1.a) imply that $\sum_i b_{ij} + c_j = 1$ while $\sum_d b_{id,j} \leq \sum_i b_{ij}$ and $\sum_d l_{id}\mathrm{w}c_j \leq c_j$. The last two inequalities and the equation preceding them imply (7), which means that D is self-sufficient in j in (3). On the other hand, if (iii) of Definition 4 is true it is possible either for the inequality or the equality to be true in (7). In the first case, the D-set has a surplus of j in (3). The second case is possible only if $\sum_d l_{id}\mathrm{w}c_j = c_j$ and $c_j > 0$, implying that $\mathrm{w} = \sum_d l_{id} = 1$. Therefore $D = n$, which means that (3) is viable according to Definition 4.

It is worth noting that system (3) corresponds to a subsistence economy or an economy with a surplus if w = 1 or w $\in$ [0,1[, respectively.

Lemma 2. *For every* w $\in$]0,1], *the matrix A corresponding to* (3) *is indecomposable.*

Proof. Due to the fact that $l_i > 0$ for each i, each good integrating the net product in (A.2), by substituting the wage in (3), virtually produces each one of the goods in the corresponding matrix A. This remark and the fact that, as shown in Section 2, in viable economies every good used as means of production produces the net product, imply that in matrix A each one of the goods is basic which, in turn, implies that A is indecomposable.

Models *S5* and *S6* are related in the following proposition.

Theorem 1. *Given a viable system* (A.2) *in which t = 1 and the corresponding system* (3):

- (i) *For each* w $\in$]0,1], (3) *and* (A.1.b) *determine a price system equal to the one determined by* (A.1.b) and (A.2) *for* $r = r$(w), *also a profit rate equal to* r(w).
- (ii) *For each* $r \in$ [0,R[, (A.1.b) and (A.2) *determine a price system equal to the one determined by* (3) *and* (A.1.b) *for* w = w(r).

Proof. Given that (A.2) is viable, for each w $\in$]0,1], matrix A in the corresponding system (3) is both viable and indecomposable as established respectively by lemmas 1 and 2. From these conclusions and theorems 4.B.1 and 4.C.10 by Takayama (1985: 372, 388), it follows that for each w* $\in$]0,1] there is only one solution (r*, p) ≥ 0 to (3). Moreover, p > 0 and p is unique up to a scalar multiple. Together with (A.1.b), this system of relative prices determines a unique system of prices p* measured with the real income. Thus, for each w* $\in$]0,1] there is only one set of variables {r*, w*, p*} that satisfies (3) and (A.1.b) such that r* ≥ 0 and p* > 0.

On the other hand, according to Theorem A.1, for $r = r(\mathrm{w}^*)$, the price vector p satisfying (A.3) is unique and strictly positive. Together with equation (A.8.b), p determines a unique wage $\mathrm{w}^* = \mathrm{w}[r(\mathrm{w}^*)]$. Then, multiplying p by w* we obtain the price system p' determined by (A.1.b) and (A.2). Furthermore, given that $\mathrm{w}^* > 0$ each price is strictly positive. According to Proposition 1, the set of variables $\{r(\mathrm{w}^*), \mathrm{w}^*, \mathrm{p}'\}$ also satisfy (3). Therefore, we have:

$$\{r^*, \mathrm{w}^*, \mathrm{p}^*\} = \{r(\mathrm{w}^*), \mathrm{w}^*, \mathrm{p}'\}$$

proving (i). The proof of (ii) follows from the previous equation and from the fact that, for every $r \in [0,R[$, there is a $\mathrm{w}^* \in]0,1]$ such that $r = r(\mathrm{w}^*)$.

We will refer to these results by saying that models *S5* and *S6* are equivalent regarding price determination.[8] Hence, we can formulate the following conclusion:

Proposition 2. *When t = 1, for each* $\mathrm{w} \in]0,1]$ *every viable system of type* (A.2) *is equivalent, regarding price determination, to a system of type* (4.b) *in which the workers receive as payment the wage goods specified by matrix U.*

To illustrate the results from this section, let us consider the following matrices:

$$\text{a)} \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \qquad \text{c)} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \tag{8}$$

We assume that they represent quantities of goods consumed as means of production and also that in each one of the corresponding economies $l_1 = 1/4$ and $l_2 = 3/4$. It follows from (2.b) that $c_a = (1/2,1/2)$, $c_b = (0,1)$ and $c_c = (1/2,1/2)$. Let $\mathrm{w}_a = 1$, $\mathrm{w}_b = 1/4$ and $\mathrm{w}_c = 1/8$. Then, after calculating in each case the coefficients of matrix U as indicated by (5), result the following matrices:

$$U_a = \begin{bmatrix} 1/8 & 1/8 \\ 3/8 & 3/8 \end{bmatrix} \quad U_b = \begin{bmatrix} 0 & 1/16 \\ 0 & 3/16 \end{bmatrix} \quad U_c = \begin{bmatrix} 1/64 & 1/64 \\ 3/64 & 3/64 \end{bmatrix}$$

In turn, substituting in (1.a) in each case d_{ij} for $l_i \mathrm{w} c_j$ and simplifying yields the following virtual methods of production:

$$A_a = \begin{bmatrix} 1/8 & 5/8 \\ 7/8 & 3/8 \end{bmatrix} \quad A_b = \begin{bmatrix} 1/2 & 1/16 \\ 1/2 & 3/16 \end{bmatrix} \quad A_c = \begin{bmatrix} 33/64 & 1/64 \\ 3/64 & 35/64 \end{bmatrix}$$

As may be observed, in each case, we have a viable and indecomposable matrix. The first one corresponds to a subsistence economy because workers receive the whole real income. In the other three cases, they receive less than that and, for this reason, each one of these matrices corresponds to an economy with a surplus. In the Appendix A.2, we present a numerical example to illustrate Theorem 1.

4. An Effect on The Price System Due to Wages Paid in Value

In this section, we consider further some results established in the previous one in order to highlight the fact that the passage from wages paid in kind to wages paid in value has an effect on the price system. To this end, we use the notation T_{S4} for the technical matrix corresponding to a viable system of type (A.2) in which, for a given level of $\mathrm{w} \in]0, 1]$, workers spend their whole revenue so that, for each i:

$$l_i \mathrm{w} = \sum_j g_{ij} \mathrm{p}_j \tag{9}$$

[8] It is worth remarking that (4.b) permits to build a single homothetic commodity for each $\mathrm{w} \in]0,1]$, although the methods of production in these systems are virtual ones. About standard commodities, see Benítez (1986).

We also define another two related matrices. The three matrices are:

$$T_{S1} = [B_{S1}\ D_{S1}\ \ 0\ E_{S1}] \qquad T_{S4} = [B_{S4}\ 0\ G_{S4}\ E_{S4}] \qquad T_{S6} = [B_{S6}\ U\ 0\ E_{S6}]$$

We assume that:

$$\text{a)}\quad B_{S1} = B_{S4} = B_{S6} \qquad \text{b)}\, E_{S1} = E_{S4} = E_{S6} \qquad \text{d)}\, D_{S1} = G_{S4} \tag{10}$$

also that, for each *i*, U_i is determined by (5) for the given level of w. In this context, T_{S1} and T_{S4} differ only by the fact that the wage goods received as payment in kind in the first technique are bought by the workers with their salaries in the second one. On the other hand, with the exception of very rare cases, we have $G_{S4} \neq U$. Thus, excluding these exceptions from now on, we may write:

$$C_{S1} = C_{S4} \neq C_{S6}$$

The three technical matrices will help us to visualize the following conclusions.

Proposition 3. *Given the techniques T_{S1} and T_{S4}, when a change is made from wages paid in kind to wages paid in value*:

(i) *The system of prices measured with the net product passes from being determined by (A.1.b) and a system of type (4.b), whose solution may not be unique, to be determined by (A.1.b) and a system of type (A.2) admitting only one solution.*

(ii) *The effect of this change on the price system is the same as the one resulting from substituting T_{S1} for T_{S6}, normally different from T_{S1}.*

The validity of the proposition is due, on the one hand, to the fact that system (4.b) in which matrix A is determined by (1.a) for a given T_{S1}, does not have necessarily a unique solution. On the other hand, it is due to Theorem 1 and Proposition 2. We will now illustrate these arguments with an example.

Let us consider the system of type (A.2) whose matrix B is equal to (8.c), the quantities of labor and the level of w are those indicated for this case in the numerical examples in Section 3. As shown in the Appendix A.2, (A.2) and the corresponding system (3) determine the same solution $r = 7/9$ and $p_2 = 3p_1$. The last result and U_c imply that $\left(\frac{1}{64}\right)p_1 + \left(\frac{1}{64}\right)p_2 = \left(\frac{1}{16}\right)p_1$ while $\left(\frac{3}{64}\right)p_1 + \left(\frac{3}{64}\right)p_2 = \left(\frac{1}{16}\right)p_2$. Therefore, workers in branch 1 may spend their whole revenue buying 1/16 of the first good and those in branch 2 buying 1/16 of the second good. If they do so, we have:

$$D_{S1} = G_{S4} = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/16 \end{bmatrix} \qquad A_{S1} = C_{S1} = C_{S4} = \begin{bmatrix} 9/16 & 0 \\ 0 & 9/16 \end{bmatrix}$$

It is important to underscore that the one and only system (4.b) where $A = A_{S1}$ (which corresponds to T_{S1}) is satisfied by the solution already indicated for (A.2) corresponding to T_{S4}. However, although both systems determine the same value of *r*, contrarily to the second system, the first one may be verified with any positive price system.

In this manner, T_{S1} and T_{S4} are equivalent from the perspective of the *PCMC*, but not regarding price determination. T_{S4} and T_{S6} are equivalent from the perspective of price determination but not regarding the *PCMC*. The first and the third technique are not equivalent from any of these perspectives. Nevertheless, when we pass from wages paid in kind to wages paid in value there is a virtual substitution of the first for the third technique. Therefore, we can formulate the following conclusion regarding the articulation of real and value variables in Sraffa's theory.

Proposition 4. *When wages are paid in value, prices are not always determined by the PCMC that actually takes place in the economy, represented by C_{S1}, but they are always determined by the virtual PCMC represented by C_{S6}. The two corresponding techniques, respectively T_{S1} and T_{S6}, are related by equations* (5), (9), (10.a) *and* (10.b).

It is important to remark that in the preceding analysis matrix U may be replaced by any wage goods matrix D verifying (6) and such that the resulting matrix A is both viable and indecomposable.

5. Concluding Comments

Sraffa dedicates most of his book to*S3* and only a few pages to *S1* and *S2*. As he does not explain the cause of this disproportion, it seems that he preferred the form of payment characteristic of the marginal theory of value and distribution to facilitate comparing his results with those of this school, according to a project announced in Sraffa (1960: vi).[9] Independently of this, the few pages dedicated to *S1* and *S2* are sufficient to document his awareness of the need to study the *PCMC* considering several different forms of wage payment. On this regard, it is important to remember that he considers *S2* as the most appropriate way to treat wages, although he thought (not completely right in this particular point) that it is possible to establish with *S2* the same results valid for *S3*.

Circularity, which is one of the distinctive traits of the *PCMC*, reaches its most complete expression in *S1-A*. Indeed, each price determines every price (given that every good is basic) and all goods produced are consumed in production. Starting from *S1-B*, circularity admits some exceptions due to the appearance of non-basic goods and, later, in *S2*, wage goods partially stop being means of production and they loss completely that condition in *S3*, *S4* and*S5*. In *S6* it is possible to observe again the same degree of circularity present in the first model because every good is virtually basic and, when $w = 1$, all goods produced are virtually consumed in production. This is so even if in matrix *A*, as in (8.c), no good enters in the production of another good. Therefore, in addition to the direct and indirect forms of production, it is possible to distinguish a virtual participation of each good in the production of every good through income distribution and, also, to define the corresponding virtual methods of production. It is worth adding that this basic dimension of goods (in the case of wage goods) in models different from*S1-A* was not ignored by Sraffa (1960:10),[10] although in *S6* it is realized in a form different from the one he foresaw.

In *S1* wage goods are part of the means of production, as in von Neuman (1945), in *S3* the methods of production are similar to those presented in Dmitriev (1974) and Leontief (1941), and in *S2* a combination is made of the two preceding forms. The model more frequently studied, *S3*, is equivalent (with the mediation of (A.6)) to *S5* and *S6*. The last one combines distinctive features from the three models studied by Sraffa and constitutes, due to this fact and to what has been already said about circularity, the most complete representation of the*PCMC* among the six models.

Finally, we will conclude by pointing out the main contributions contained in each section that, to the best of our knowledge, have not been previously published. Section 2: Definitions 1 and 4; Section 3: propositions 2 and 3, lemmas 1 and 2, Theorem 1; Section 5: propositions 3 and 4.

APPENDIX

A.1 Model *S5*

Sraffa (1960: 10) considers the treatment of the wage in *S2* as the most appropriate and, on the other hand, for the most part in his work, wages are paid entirely in value at the end of production.[11] Combining the two forms of payment, in this appendix we assume that wages are paid entirely in value, a fraction $t \in [0,1]$ being paid on the first date and the rest $(1 - t)$ on the second one. In this manner, the wage cost in the economy is $tw(1 + r) + (1 - t)w = w(1 + tr)$. It must be noted that model *S5*, so established, includes *S3* and *S4* as particular cases, as well as all the intermediate forms of payment between them. Moreover, $d_{ij} = 0$ $\forall$ (i,j) and, therefore, the first of the following equations is true:

$$\text{a) } a_{ij} = b_{ij} \; \forall \quad (i,j) \qquad \text{b) } \sum_j c_j p_j = 1 \tag{A.1}$$

[9] "It is, however, a peculiar feature of the set of propositions now published that, although they do not enter into any discussion of the marginal theory of value and distribution, they have nevertheless been designed to serve as the basis for a critique of that theory." According to Negishi (1985: 73-76) the assumption that wages are not paid with the past but with the current product is proper to the Post-Walrasian school while advanced wages are more compatible with Marxist theory.

[10] "Necessaries however are essentially basic and if they are prevented from exerting their influence on prices and profits under that label, they must do so in devious ways (e. g. by setting a limit below which the wage cannot fall; a limit which will itself fall with any improvement in the methods of production of necessaries, carrying with it a rise in the rate of profits and a change on the prices of other products)."

[11] "In any case the discussion which follows [referred to *S3*] can easily be adapted to the more appropriate, if unconventional, interpretation of the wage suggested above [*S2*]."

The second equation is also true adopting the value of the real income as unit of measurement.[12] Thus, it is possible to write the production equations in the following form:

$$\sum_j a_{ij}\, p_j\,(1+r) + l_i w(1+tr) = p_i \qquad i = 1,2,\dots,n \qquad (A.2)$$

To solve the system formed by (A.1.b) and (A.2) it is useful to calculate in the first place prices in wage units and afterwards, as indicated below, to calculate prices and wages measured with the real income. Indeed, this allows us to consider initially neither the variable w nor the second equation, having to solve only the following system:

$$\sum_j a_{ij}\, p_j\,(1+r) + l_i(1+tr) = p_i \qquad i = 1,2,\dots,n \qquad (A.3)$$

Regarding this system, the following propositions are true.

Theorem A.1. *If A is a surplus ESRS there is an interval* $[0,R[$ *such that:*

(i) *R is independent of t and* $0 < R < +\infty$.
(ii) *For each* $r \in [0,R[$, *the solution of* (A.3) *is unique and strictly positive.*
(iii) $p_j(r)$ *is a monotonous increasing function for every j.*
(iv) *At least one price tends to infinity when r tends to R.*
(v) *For each* $r \in [0,R[$, *the quotient* $p_i(r)/p_j(r)$ *is independent of t* $\forall$ *(i,j).*

Proof. See Benítez (2009).

According to the Appendix of the paper just quoted, after multiplying each one of the n equations of (A.3) by the quotient $(1 + r)/(1 + tr)$ results:

$$\sum_j a_{ij}\, p_j \left[\frac{(1+r)}{1+tr}\right](1+r) + l_i(1+r) = p_i\,\frac{(1\,+\,r)}{1\,+\,tr} \qquad i = 1,2,\dots,n \qquad (A.4)$$

Let

$$\mathrm{p}_j = p_j(1 + r)/(1 + tr) \qquad i = 1,2,\dots,n \qquad (A.5)$$

$\Rightarrow$

$$p_j = \mathrm{p}_j(1 + tr)/(1 + r) \qquad i = 1,2,\dots,n \qquad (A.6)$$

Substituting in (A.4), for each j, the quotient $p_j(1 + r)/(1 + tr)$ for the left side of the corresponding equation (A.5) yields:

$$\sum_j a_{ij}\, \mathrm{p}_j\,(1+r) + l_i(1+r) = \mathrm{p}_i \qquad i = 1,2,\dots,n \qquad (A.7)$$

System (A.6) permits to calculate prices in wage units, for every $t \in [0,1[$, from their values when $t = 1$, determined by (A.7). Once they are known, in order to calculate prices measured with the real income, it is enough to multiply prices in wage units by the corresponding value of w, which may be established by means of the first of the following equations:

$$\text{a) } w = 1/\sum_j c_j\, p_j \qquad \text{b) } \mathrm{w} = 1/\sum_j c_j\, \mathrm{p}_j \qquad (A.8)$$

In both equations prices are measured in wage units and, in the second one, w represents the wage when it is entirely advanced. Summing up the n equations of (A.2), when $t = 1$, yields

[12] Equation (A.1.a) implies thatin this model the right sides of (2.a) and (2.b) are equal.

$\sum_j \sum_i a_{ij}$ $p_j(1+r) + \sum_i l_i w(1+r) = \sum_i p_i$. It is useful to remember that $\sum_i l_i = 1$, also to remark that from (2.b) it follows the equality $\sum_i p_i = \sum_j \sum_i a_{ij}$ p_j + $\sum_i c_i p_i$. By substituting the left side of the last two equations for the corresponding right side in the sum of the n equations of (A.2) we obtain $\sum_j \sum_i a_{ij}$ $p_j(1+r) + w(1+r) = \sum_j \sum_i a_{ij}$ p_j + $\sum_i c_i p_i$. This result and (A.1.b) imply that $\sum_j \sum_i a_{ij}$ $p_j r + w(1+r) = 1$, which means that w = 1 when $r = 0$. On the other hand, according to Section 2 every good used as means of production produces the net product, this fact and (iv) of Theorem A.1 imply that w = 0 when $r = R$. In these conditions, we can define the first of following functions:

$$w(r): [0,R[\rightarrow]0,1] \qquad\qquad r(w):]0,1] \rightarrow [0,R[$$

associating to each r the value of w determined by (A.8.b). It follows from this equation and (iii) of Theorem A.1 that w(r) is a monotonous decreasing function. Thus, we can also define its inverse function (the second of the previous functions) associating to each w the value of r for which w = w(r).

Finally, (A.8.a) and (A.8.b) permit to calculate w for every $t \in [0,1[$ from its value when $t = 1$. Indeed, substituting each price in the second equation for its equivalent in (A.5) yields:

$$w = 1/\sum_j \frac{c_j p_j (1+r)}{1+tr} = 1/\left[\frac{(1+r)}{1+tr}\right]\sum_j c_j p_j \Rightarrow w(1+r)/(1+tr) = 1/\sum_j c_j p_j$$

This result and (A.8.a) imply that:

$$w = w(1+r)/(1+tr) \qquad (A.9)$$

This equation allows us to observe that, for every $r \in]0,R[$, the wage is a monotonous decreasing function of t while prices increase monotonously, according to (A.6).

A.2 A Numerical Example for Theorem 1

Let B = (8.c), $l_1 = 1/4$ and $l_2 = 3/4$. For any w $\in$]0,1], we have the following system of type (A.2):

$$\begin{aligned} &[(1/2)p_1 + (1/4)w](1+r) = p_1 \\ &[(1/2)p_2 + (3/4)w](1+r) = p_2 \end{aligned} \qquad (A.10)$$

If w = 1/8, it is possible to write (A.10) as the first of the following systems. After ordering by columns and simplifying we obtain the second one:

$$\begin{aligned} &[(1/2)p_1 + 1/32](1+r) = p_1 \quad [(1/2) - 1/2r]\, p_1(1+r) = (1/32)(1+r) \\ &[(1/2)p_2 + 3/32](1+r) = p_2 \quad [(1/2) - 1/2r]\, p_2(1+r) = (3/32)(1+r) \end{aligned}$$

Dividing each side of the second equation in the second system by the corresponding side of the first equation, we obtain $p_2/p_1 = 3$. Now, substituting w for $(1/8)[(1/2)p_1 + (1/2)p_2]$ in (A.10) and adding up the two equations yields:

$$[(1/2)p_1 + (1/2)p_2](1+r) + (1/8)[(1/2)p_1 + (1/2)p_2](1+r) = p_1 + p_2$$

Dividing the two sides by $(1/2)(p_1 + p_2)$ gives $(1+r) + (1/8)(1+r) = 2$ so that $9 + 9r = 16$. Therefore, $r = 7/9$. On the other hand, when w = 1/8, matrix $A = A_c$ in the system (4.b) corresponding to (A.10). Hence, this system is:

$$\begin{aligned} &[(33/64)p_1 + (1/64)p_2](1+r) = p_1 \\ &[(3/64)p_1 + (35/64)p_2](1+r) = p_2 \end{aligned} \qquad (A.11)$$

Substituting r for 7/9 in both equations, p_2 for $3p_1$ in the first equation and p_1 for $(1/3)p_2$ in the second one yields, in the first equation $(36/64)p_1(16/9) = p_1 \Rightarrow (4/64)p_1(16) = p_1$ and in the second equation $(36/64)\, p_2(16/9) = p_2 \Rightarrow (4/64)p_2(16) = p_2$. These results verify that the solution determined by (A.10) also satisfies (A.11). On the other hand, the fact that A_c is indecomposable imply, according to theorems

4.B.1 and 4.C.10 by Takayama (1985: 372, 388), that the solution $(r;p) \geq 0$ determined by (A.11) is unique, $p > 0$ and p is unique up to a scalar multiple. Therefore, (A.10) and (A.11) determine the same exchange rate between the two goods and the same profit rate for the given level of w.

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