Elicitating preferences in Game Theory: A fuzzy approach

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ABSTRACT

This paper proposes a systematic methodology to elicitate preferences at the Game Theory payoff matrix reducing the difficulty of coping only with numerical variables. Moreover, sometimes the fact is that it is not possible to delimit the preferences of decision makers by exclusionary intervals. The authors believe that it can facilitate the process by using a fuzzy approach and linguistic evaluation to deal with the complexity, imprecision and uncertainty of the context. Another benefit is that a more realistic scenario can be built excluding the common deterministic approach. An example, using the dominated strategies concept, is given to illustrate the proposal.

KEYWORDS: PREFERENCE ELICITATION; GAME THEORY; FUZZY THEORY

1. Introduction

The utility theory, conceived by Von Neumann and Morgenstern, is an indispensable tool for knowledge areas as Decision Theory, Multicriteria Decision, Microeconomics and Game Theory. The need for the concept originated with the desire to treat problems of decision under risk or uncertainty. For example, in an important literature in the area, already showed a great concern in guiding the use of such a theory and concluded that despite being one of the pillars of Game Theory, the utility is applied and can be understood in other contexts (Luce and Raiffa, 1957),

Among the set of multicriteria decision-making methods, the MAUT - multiattribute utility theory - is the only that has a well structured protocol and is supported by an axiomatic structure associated to determining preferences (Almeida, 2011) states that,. This process is called elicitation or eduction. According to the author, in the other methods the decision maker only specifies parameters by an ad-hoc way.

There are problems regarding the way to express preferences by utility functions without regard concepts as ordinalidade and cardinality (Souza et al, 2008). These same authors have highlighted the lack of guidance in basic microeconomics books and game theory in relation to preferences eduction.

In applying game theory to model a practical problem, there is an intrinsic difficulty in the process of extracting the preferences of decision makers and represent them, since they contain information that represents the imprecision of real situations (Maeda, 2003). Still, the author notes that, for this type of situation, you can use fuzzy numbers.

This work, therefore, will bring a proposal for systematic preferences eduction on payoff matrix using fundamentals of fuzzy set theory in order to resolve problems and facilitate the process in situations of uncertainty and imprecision in the context of game theory. It will be also presented an application of the concept of dominance to a fuzzy payoff matrix.

2. Utility function

The microeconomic theory is based on the idea that individuals are able to establish preferences between goods, and from them, are defined utility functions, i.e. a real function that preserves and expresses the order of the choices of the agent, and in some cases quantify the gain or loss of utility. Thus, in general, there are two ways to set preferences for a rational agent from their utility functions according to microeconomic perspective: the ordinal utility and cardinal utility.

Defining A as the set of alternatives available to an individual, and 'a' and 'b' as any two elements of the set A. If the agent reveals that a≻ b, then it is possible to define a utility function" "u" such that u(a)>u(b). So when the only interest is to express the order of individual preferences, the function U (.) should use an ordinal level of measurement.

Therefore, the magnitude of the utility of the options does not matter, only being possible as stated in section 2, express the relationships as "better", "worse" or "as good as". Moreover, the preferences expressed as cardinal are measured on at least interval scale, and thus are at a higher hierarchical level of sophistication than the ordinal utilities.

Now, for example, any given three components 'a', 'b' and 'c' of the set A, so that the a≻b≻c, it is possible to quantify the variation of utility, ie well as preserving the ordering of preferences, cardinal utility also enables measuring how many times D(a,c) = u(a)- (c) is greater than D(a, b) = u(a)-u (b).

Moreover, from the perspective of Decision Theory, it is defined "u" as a utility function if (Campello de Souza, 2007):

a) $u: P^* \to R$, ie., for every distribution P ϵ P^* represents a real number $u(P)$.
b) These real numbers assigned to preserve order in the sense that

$$P \gtrsim Q \leftrightarrow u(P) \geq u(Q)$$

c) Linearity exists:

$$u[\lambda P + (1-\lambda) Q] = \lambda u(P) + (1-\lambda) u(Q)$$

In the utility theory, one can avoid inconsistencies in the structuring of the problem with making the decision maker is consistent with their preferences in order to satisfy the axioms of the theory (Almeida, 2011). Thus, we have that the utility function of this decision maker should be in line with the axiomatic basis. Since this is given by the following axioms (Keeney and Raiffa, 1976; Raiffa, 1968):

d) Ordenability: given the alternatives A and B, it can be said that APB (A is preferable to B), or AIB (A is indifferent B) or (BPA B is preferred to A);
e) Transitivity: for preference, if APB and BPC, then APC; indifference to, if AIB and BIC, then AIC;
f) Dominance: if APB, then there p, 0 ≤ "p ≤ 1", such that for any C: [A, p, C p-1] P [B p, C p-1]; valid for I in place of indifference preferably P;
g) Archimedean: if APBPC, then there q, 0 <q <p <1, such that [A, p, C p-1] PBP [A, q C, 1-q].

In general, microeconomic theory is based on the idea that individuals are able to establish preferences among alternatives, and from them, are defined utility functions, ie a real function that preserves and expresses the order of the choices of the agent, and in some cases quantify the gains and losses of utility. Thus, in general, there are two ways to define the preferences of a rational agent from their utility functions, they are: the ordinal utility and cardinal utility.

Defining A as the set of alternatives available to an individual, and 'a' and 'b' as any two elements of the set A. If the agent reveals that a ≻ b, then it is possible to define a utility function u such that u(a)> u(b). So when the only interest is to express the order of individual preferences, the function U (.) should use an ordinal level of measurement.

So when what you have is an ordinality paradigm, the magnitude of the usefulness of the options does not matter, just being able to express the relationship as "best," "worst" or "as good as". Moreover, the preferences expressed as cardinal are measured on at least interval scale, and thus are at a higher hierarchical level of sophistication than the ordinal utilities.

Now, for example, any given three components 'a', 'b' and 'c' of the set A, so that the a≻b e b≻c, it is possible to quantify the variation of utility, ie well as preserving the ordering of preferences, cardinal utility also enables measuring how many times D (a, c) = u (a) - u (c) is greater than D (a, b) = u (a) - u (b).

Therefore, the determination of the utility function is associated with confirmation of the relationship between the structure and the structure shown axiomatic decision maker's preferences (Raiffa and Keeney, 1976). The next subsection will deal with ways to obtain a given function.

2.1 Construction of Utility Function

It is possible to obtain a utility function that satisfies the axioms presented earlier, a given individual in a given context? The answer to this question is maybe and feature a series of attempts to clarify the issue, presenting the attempts of many experts and some associated problems (Luce and Raiffa, 1957).

There is a way to determine the utility function through the procedure described below (Nogee and Mosteller, 1951):

Consider a lottery between the consequences of greater and lesser preference x_n e x_1: $[x_n, p; x_1, 1-p]$. If decision maker have to declare their preferences between this lottery and each result x_i, then the decision maker is offered two options, one option called certainty, x_i, and another called option risk $[x_n, p; x_1, 1-p]$. For each indifference between x_i and the lottery, you get a probability p_i, which represents the probability for which the decision maker is indifferent between the lottery $[x_n, p; x_1, 1-p]$ and the result x_i. Whereas the decision maker is consistent there is p_n=1 e $p_1 = 0$. The probabilities obtained are related to each alternative value or range under examination result and will be their own utilities.

The utility function for a set of consequences or alternatives can be obtained by two methods: direct evaluation and estimation of the utility function (Almeida, 2011). The process of direct evaluation of the usefulness consists in obtaining direct utility value for each alternative or result in the problem of interest. According to the same author, this procedure is limited to problems with few alternatives or consequences. Furthermore, the estimation procedure of the utility function is indicated in the cases where there is a larger number of alternatives or consequences.

An obvious difficulty which may be mentioned from attempts to determine the utility function is to search for the confirmation of the theory works for a multitude of points from a finite number of assuming it is linear. More clearly, the procedures often prove the axioms for a finite number of points and this is generalized to the infinite points examined (Luce and Raifa, 1957).

Yet another problem that arises is the presence of intransitivities from the preferences elicitation, which is inconsistent with the axioms. Respondents inevitably contain errors and random turbulence when questioned about their preferences. Finally, there is the issue of whether it is a matter whose nature is subjective and have to meet well objectives axioms (LUCE and RAIFFA, 1957).

Thus, some authors (Campello de Souza, 2007; Luce and Raiffa, 1951; Almeida, 2011) argue that the elicitation/eduction regarding monetary criteria are simpler. A questionnaire eductor for money method called overlapping tracks can be found in Campello de Souza (2007).

In general, it can be said that the task of determining the preferences from the individuals utility function is in fact something quite complex and possibly sometimes becomes unfeasible from a practical standpoint. Despite this statement, this does not invalidate the importance of the findings and decision methods that make use of the utility function (Luce and Raiffa, 1957).

Given the difficulties, a way to proceed with the determination of preferences is proposed in this work makes use of assessments and fuzzy, often used when the situation is uncertain and imprecise nature. The next section will be devoted to the topic.

3. Fuzzy Linguistic evaluation

The fuzzy set theory was proposed by Zadeh in 1965 and has been considered as a basis for linguistic approaches (or numerical) and imprecise character (Zadeh, 1965). Inaccuracy of information is one of the difficulties encountered by the decision maker for many conditions (Güngör and Arikan, 2000). In this context, the theory of fuzzy sets can represent an important tool for modeling problems where there is a difficulty in measuring the preferences of the decision maker.

Furthermore, the fuzzy analysis does not requires quantitative data from decision makers. The performances of the variables can be described in linguistic terms as "bad," "good," "very good," etc. (Güngör and Arikan 2000; Zadeh, 1965, Wang, 1997). Thus, the linguistic evaluation can be a facilitating tool for obtaining reviews of any kind, since it may be easier for the decision maker to use it instead of being asked to define precise numbers.

For linguistic variable means that their values are words or sentences in a natural or artificial language (Zadeh, 1975). According the author is entirely possible that the qualitative approach, might-have tool for system analysis too complex or poorly defined for the application of conventional quantitative techniques. He states that it is important to note that using the called extension principle, a lot of mathematical apparatus to analyze existing systems can be adapted for manipulating linguistic variables and thus be developed approximate calculations of linguistic variables that can be widely used in various applications.

There are some concepts of linguistic variables that need to be understood as the difference between compatibility and probability. For example, an assessment of whether a temperature of "30 degrees C" evaluation "hot" being 0.7 bears no relationship to the probability of a temperature of 30 ° C. The interpretation of the value 0.7 compatibility is a subjective indication of the extent of how "hot" is 30 ° C.

Furthermore, it is assumed that the linguistic variable is structured according to two rules: (i) a syntactic rule specifies how the linguistic values in a set of variables must be generated. And for this, it is assumed that the terms within the set of variables is generated by a context-free grammar, (ii) a semantic rule specifies the procedure for calculating the significance of any linguistic value. In this context, it is noted that a typical value of a linguistic variable, for example, not too cold, involves what could be called, in primers terms, for example, fresh, meaning both subjective and context dependent.

Besides the above, one may add to the terms primary, expressing the linguistic variables, or some terms as no, more, less, completely, extremely a little, quite etc.. These have the feature to change the terms of these primary attributing some intensity (Zadeh, 1975).

Another issue that must be considered is that certain variables have a numerical scale that is, such as temperature. Moreover, there are variables that do not have a well-defined variable basis, i.e. not know how to express, for example, the degree of happiness to a person as a function of physical measurements. However, what should be noted is that there how to convert numbers into words what we know as the linguistic approach (Zadeh, 1975).

Besides the above, it is necessary to drill a fuzzy number may assume many different shapes as, for example, a function trapezoidal, triangular or Gaussian (Gupta and Kaufmann, 1988).

The use of fuzzy numbers in this paper is directed to triangular membership functions. The motivation behind the use of triangular fuzzy numbers stems from its applicability to the context and presented in the simplicity of its membership function (Pedrycz, 1994).

The following section describes the triangular fuzzy numbers

3.1 Triangular Fuzzy Numbers

The triangular fuzzy numbers are referred to in the literature, for TFNs (Gupta and Kaufmann, 1988), and its representation may be seen in Figure 1. A TFN can be parameterized by the triple (a1, a2, a3), exemplified in the Figure by the same function ABC (Nguyen et al, 2008).

In the example shown in Figure 1, it can be seen that the element has 0,5 degree of membership for the function ABC.

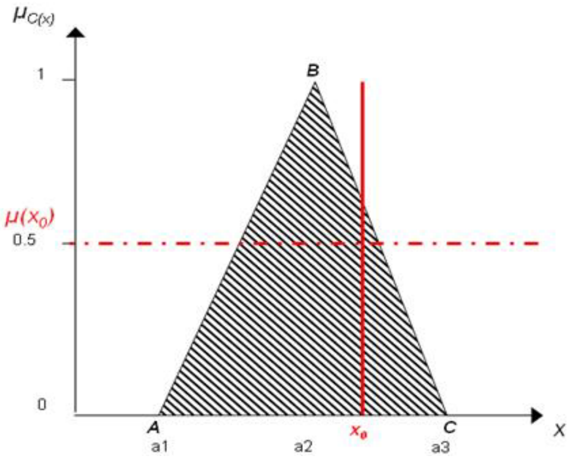

Figure 1 – Triangular function

X is defined as a set non-empty universe, and A_i is the i_{th} fuzzy subset of X such that A_i={a_1,a_2,a_3} i=1,2,3,...,n, where $a_1<a_2<a_3$ is the scale of the structure preferably being used by the maker and "n" is the number of fuzzy elements to be used in the analysis. The fuzzy set Ai in X is characterized by a membership function μ_A which associates with each element x in X a real number in the interval [0,1]: μ_A = X → [0,1] (Kaufmann e Gupta, 1988). The value of $\mu_A(x)$ represents the degree of relevance of an element x in a fuzzy subset A for each x in the X (Zadeh, 1965; Nguyen et al., 2008). In other words, the element x assumes three states: x is said belonging to A if $\mu_A(x)$=1; x does not belong to A if μA (x) = 0 and x is in A with a the degree of relevance if μA (x) 0 <μA (x) <1. A TFN can be defined by the triple A = (a1, a2, a3) whose function triangular relevance, μA (x) is defined by Equation (1) (Gupta and Kaufmann, 1988).

$$\begin{cases} 0, & x < a_1 \\ \dfrac{(x-a_1)}{(a_2-a_1)}, & a_1 \le x \le a_2 \\ \dfrac{(a_3-x)}{(a_3-a_2)}, & a_2 \le x \le a_3 \\ 0, & x > a_3 \end{cases} \qquad (1)$$

A linguistic variable may be represented by quadruple (X, LX, U, Mx), where (Driankov et al. 1993):
i. X is the variable name;
ii. LX is the set of linguistic terms can assume that X;
iii. U is the universe of discourse where are all fuzzy sets defined for the linguistic terms of the variable X;
iv. Mx is a rule that associates with each linguistic value their numerical significance in the physical domain U.
The following section presents the proposal of fuzzy payoff matrix

4. Proposed fuzzy *Payoffs matrix*

The proposal for eduction of the utility function of the payoff matrix has as main objective to translate the uncertainties relating to the judgment of the decision maker by using fuzzy linguistic variables reducing the difficulty of this treatment in evaluations of numerical variables. Furthermore, it should be noted that it is not always possible to discern the preferences of the decision maker in order to represent them by excluding intervals.

Eduction for the payoff matrix, it is proposed that, initially, the preferences are extracted through evaluations language about the object of study, where the decision maker must identify their degree of preference in relation to strategies. Assuming a game between two participants (A and B) zero sum, where there are three strategies to be evaluated, {E1, E2, E3}, each table entry must represent the utility awards for said player. However, this utility is extracted through evaluations of the decision maker language that can represent your preferences. The decision maker is presented with the opportunity to express their preferences through only the evaluations of each strategy. Thus, for each strategy it will set your preference through the following expressions: Strongly not preferable, not better, Indifferent, and Preferred Heavily preferable. The scale corresponding to these linguistic values ranging from 0 to 1, and the functions are triangular fuzzy set symmetrically along the axis "x".
Figure 2 depicts the linguistic variables for evaluating strategies.

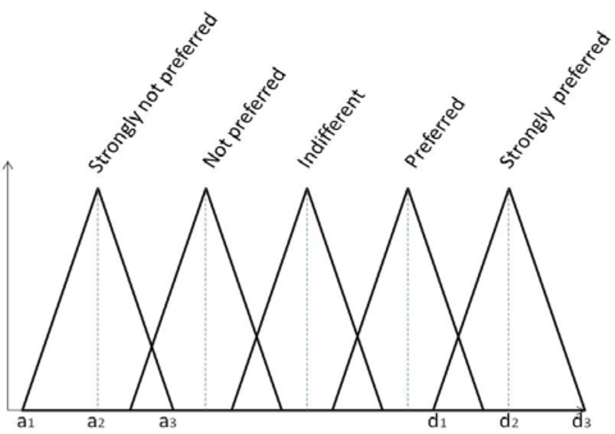

Figure 2 – Linguistic variables for preferences eduction

The proposal presented here concerns the eduction payoff matrix for players A and B, considering the fuzzy character of the decision maker's preferences.

The quadruple (X, LX, U, Mx) is defined by (Driankov et al. 1993):

i. X: preference regarding the strategy evaluated
ii. LX: {Strongly not preferable, Not preferable, Indifferent, and Strongly preferred Preferable};
iii. U: 0 to 1;
iv. Mx assigns to each linguistic value a fuzzy set whose membership function expresses its meaning.

The expected outcome of the assessment of decision maker's preferences by linguistic variables is the possibility of extracting their preferences considering the uncertainties regarding their assessments. Thus, we seek to obtain a higher degree of reality because their preferences do not need to be bounded by deterministic evaluations. The following section presents an application of the proposal presented here involving the concept of dominated strategies.

5. Application involving the concept of dominated strategies

Once defined the payoff matrix for players A and B, the proposal presented here is to use the concept of dominated strategies that says a strategy "a" is dominated by a strategy "b" if "a" is at least as good and "b" and sometimes better (Hillier 2005). If the strategy is mastered, it can be removed immediately without further considerations.

To evaluate the dominance relationship among the strategies proposed to assess the relationship preferably by using α-cuts (Gheorghe et al. 2004; 2005).

Considering the payoff matrix below (Table 1) where each player's prize A, for example, is represented by a fuzzy value (C=(c_1,c_2,c_3)), to assess if the strategy 2 is dominated by 1 we need to assess the relationship of dominance $_{E1}S_{E2}$. To do so, we must evaluate the pairwise relations: (C, F), (G, H) and (K, L). However, these pairs of actions that represent the strategies are fuzzy numbers and the proposal presented here is that they are compared by a binary relation preference based on intervals called α-cuts.

Table 1 – Payoff matrix

		Player B		
		E1	**E2**	**E3**
Player A	**E1**	C	G	K
	E2	F	H	L
	E3	D	J	M

The evaluation of relative preferably by using α-cuts intervals (Figure 3) is justified even for ease of calculation that its use provides when manipulating triangular fuzzy numbers (Gheorghe et al., 2004, Lu and Wang, 2005).

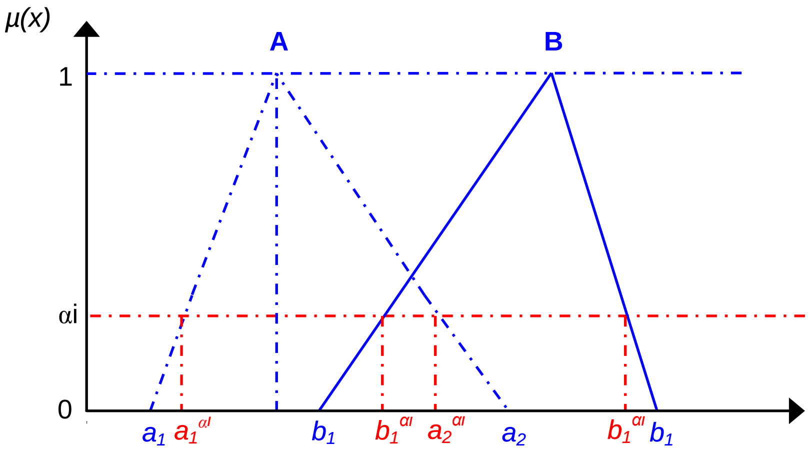

Figure 1 –α-cuts representation

Considering two triangular fuzzy numbers A and B, normalized and convex, characterized by their membership functions μ_A and μ_B: the α_i*-cut* defines the ranges $(a_1^{\alpha i}, a_2^{\alpha i})$ to A, and $(b_1^{\alpha i}, b_2^{\alpha i})$ to B, where $i = \overline{1, N}$, and N is the number *α-cuts*. Considering the triangular fuzzy number (a_1,a_2,a_3), the intervals α-cuts can be found by the following Equation (Giachetti and Young, 1997):

$$[(a_2 - a_1)\alpha + a_1, (a_2 - a_3)\alpha + a_3], \quad \forall\alpha \qquad (1)$$

For each cut-α is defined a range determined by means of Equation (1), and based on these intervals are constructed dominance indexes to represent the preference relation between strategies. As $a^{\alpha i}$ interval (Figure 3) is designed to right, the degree of dominance of A with respect to B increases (Gheorghe et al., 2004).

The indices that represent the relationship of dominance are: left αi-cut index and right αi-cut index. For each αi-cut level, an index αi-cut is defined as a function RxR in [0,1], for each pair of alternatives (a, b) represented respectively by the fuzzy numbers A and B as follows:

$$S^{\alpha i}(a^{\alpha i}, b^{\alpha i}) = S_l^{\alpha i}(a^{\alpha i}, b^{\alpha i}) + S_r^{\alpha i}(a^{\alpha i}, b^{\alpha i}) \qquad (2)$$

For each αi-cut level, an index on the left $S_l^{\alpha i}(a^{\alpha i}, b^{\alpha i})$ is defined as a function RxR in [0,1], where R is the set os real numbers.

$$S_l^{\alpha i}(a^{\alpha i}, b^{\alpha i}) = \begin{cases} 0, & a_2^{\alpha i} < b_1^{\alpha i}, \\ \dfrac{a_2^{\alpha i} - b_1^{\alpha i}}{a_2^{\alpha i} - a_1^{\alpha i}}, & a_1^{\alpha i} < b_1^{\alpha i} \le a_2^{\alpha i} \\ 1, & a_1^{\alpha i} \ge b_1^{\alpha i}. \end{cases} \quad (3)$$

For each αi-cut level, an índex on the right, $S_r^{\alpha i}(a^{\alpha i}, b^{\alpha i})$ is defined as a function RxR in [0,1], where R is the set os real numbers.

$$S_r^{\alpha i}(a^{\alpha i}, b^{\alpha i}) = \begin{cases} 0, & a_2^{\alpha i} < b_1^{\alpha i}, \\ \dfrac{a_2^{\alpha i} - b_1^{\alpha i}}{b_2^{\alpha i} - b_1^{\alpha i}}, & b_1^{\alpha i} \le a_2^{\alpha i} < \\ 1, & a_2^{\alpha i} \ge b_2^{\alpha i}. \end{cases} \quad (4)$$

Equation 2 gives the dominance relationship between any two actions, such as the actions C and F. Once all the E1 line dominates the strategy E2 line, it can be said that it is preferable to one another. And so, delete the second strategy payoff matrix.

6. Final Remarks

This work aimed to make a proposal to use fuzzy numbers in the payoff matrix of Game Theory. This proposal proved to be of great interest since the eduction or elicitation process of preferences by the utility function is considered quite complex. For example, while it is of great importance in the field, it is difficult to obtain since it implies in agreement with a well defined axiomatic base (Luce and Raiffa, 1957). One could say then that in complex situations and uncertainty and imprecision, using this fuzzy concept can come to systematize the problem of determining preferences of those involved.

The presented model is suitable for situations of uncertainty or difficulty in assessing strategies by players, without, however, considering the situations of uncertainty associated with the consequences of strategies.

To illustrate the proposal in question was presented a numerical application of the concept of fuzzy dominance to the matrix. For this, use has been made of the dominance relationship through the use of α-cuts. It is intended in future to apply other concepts proposed matrix as Nash equilibrium in pure and mixed strategy.

Finally, it is intended, from the completion of the work, encourage other studies involving game theory to fuzzy numbers, opening new possibilities in the field of conflict analysis, which repeatedly presents highly subjective.

Referências

Almeida ATA, 2011. The knowledge and use methods of multicriteria decision support: University Press.

Campello de Souza FM, 2007. Rational decisions in situations of uncertainty, 2nd edition, Recife.

Driankov D, Hellendoorn H, Reinfrank M, 1993. An introduction to fuzzy control. Berlin: Springer-Verlag.

Güngör Z, Arikan F, 2000. A fuzzy outranking method in energy policy planning, Fuzzy sets and Systems, 114(1): 115-122.

Gheorghe R, Bufardi A, Xirouchakis P, 2004. Construction of a two-parameters fuzzy outranking relation from fuzzy evaluations, Fuzzy sets and Systems, 143: 391-412.

Gheorghe R, Bufardi A, Xirouchakis P, 2005. Construction of global fuzzy preference structures from two-parameters single-criterion fuzzy outranking relations, Fuzzy sets and Systems, 153: 303-330.

Giachetti RE, Young RE, 1997. A parametric representation of fuzzy numbers and their arithmetic operators, Fuzzy Sets and Systems, 91: 185-202.

Hillier FS, Lieberman GJ, 2005. Introduction operations research: The McGraw-Hill Companies, Inc.

Kaufmann A, Gupta MM, 1998. Fuzzy Mathematical Models in Engineering and Management Science. Usa: Elsevier Science Publishers.

Keeney R, Raiffa H, 1976. Decisions with multiple objectives: preferences and value tradeoffs: New York.

Lu H-W, Wang CB, 2005. An Index for Ranking Fuzzy Numbers by Belief Feature. Information and Management Sciences, 16(3): 57-70.

Luce RD, Raiffa H, 1957. Games and Decisions – Introduction and Critical Survey, New York: Willey.

Maeda T, 2003. On characterization of equilibrium strategy of two-person zero-sum games with fuzzy payoffs, Fuzzy Sets and Systems, 139: 283–296.

Mosteller F, Nogee P, 1951. An experimental measurement of utility, Journal of Political Economy, 59(5): 371-404.

Nguyen TH; Shebab T, Gao Z, 2008. Selecting an architecture engineering team by using fuzzy set theory Engineering, Construction and Architectural Management, 15(3): 282-298.

Pedrycz W, 1994. Why triangular membership functions? Fuzzy Sets and Systems, 64: 21-30.

Raiffa H, 1968. Decision Analysis: Addison-Wesley.

Souza FC, Silva MM, Sousa FR, 2008. A discussion on the use of ordinal and cardinal utility: Comments practical books in basic microeconomics and game theory In: CLAIO XIV Latin Ibero-American Congress on Operations Research – Book of Extended Abstracts.

Wang J, 1997. A fuzzy outranking method for conceptual design evaluation, International Journal of Production Research, 35: 995–2010.

Zadeh LA, 1965. Fuzzy sets, Information and Control, 8: 338-353.

Zadeh LA, 1975. The concept of a linguistic variable and its application to approximate reasoning, Information Sciences, 8: 199–249 (I), 301–357(II).